Logarithmic behavior of some combinatorial sequences

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February 2, 2008

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Abstract

Two general methods for establishing the logarithmic behavior of recursively defined sequences of real numbers are presented. One is the interlacing method, and the other one is based on calculus. Both methods are used to prove logarithmic behavior of some combinatorially relevant sequences, such as Motzkin and Schröder numbers, sequences of values of some classic orthogonal polynomials, and many others. The calculus method extends also to two- (or more-) indexed sequences.

Keywords: log-concavity, log-convexity, special combinatorial numbers, calculus, integer sequences

AMS subject classifications: 05A20, 05A16, 05E35, 11B83, 11B37, 11B39, 11B68

1 Introduction

Let $a(n)$, $n \geq 0$, be a sequence of positive real numbers. We want to examine the rate of growth of this sequence, i.e. to examine whether the quotient $\frac{a(n)}{a(n-1)}$ decreases, increases or remains constant. In other words, we want to see whether the sequence is **log-concave**, i.e. $a(n)^2 \geq a(n-1)a(n+1)$, **log-convex**, i.e. $a(n)^2 \leq a(n-1)a(n+1)$, or **log-straight** (or **geometric**), i.e. $a(n)^2 = a(n-1)a(n+1)$ for all $n \geq 1$. Under log-behavior we also sometimes include **log-Fibonacci** behavior, meaning $sign[a(n)^2 - a(n-1)a(n+1)] = sign(-1)^n$ (or $(-1)^{n+1}$). It is of great interest, especially in combinatorics, as it can be seen from many examples in [25], to know the log-behavior of a given sequence. It is, in fact, just one instance of the whole paradigm of "positivity questions" ([28]).

If $a(n)$ has a combinatorial meaning it would be ideally to provide a combinatorial proof of its log-behavior. For example, if we want to prove that $a(n)$ is log-convex and if we know that $a(n) = |S(n)|$, where $S(n)$ is a certain finite set, then we would like to find an injection $S(n) \times S(n) \to S(n-1) \times S(n+1)$, or a surjection $S(n-1) \times S(n+1) \to S(n) \times S(n)$, and similarly for log-concavity. It is usually a hard task to find such a (natural) injection or surjection. Still, examples of this type include binomial coefficients, Motzkin numbers ([8]) and permutations with a prescribed number of runs ([6]). Of course, the explicit formulae give another possibility to prove results of this type, but they are rarely on disposal. Instead, other methods for proving such inequalities have been developed, e.g. see [22], [25], [7] or [5].

In this paper besides using old methods to prove some new results on log-behavior, we shall also introduce some new methods and use them to prove log-behavior of certain interesting combinatorial sequences, and apply this method to other sequences, the most prominent example being values of classical orthogonal polynomials.

2 Log-behavior of some sequences using known results

Let us quote some known results and apply them to examine the log-behavior of certain combinatorial and other sequences.

Lemma 2.1 (Newton's lemma)

Let $P(x) = \sum_{k=0}^{n} a_k x^k$ be a real polynomial whose all roots are real numbers. Then the coefficients of $P(x)$ are log-concave, i.e. $a_k^2 \geq a_{k-1} a_{k+1}$, $k = 1, \ldots, n-1$. Moreover, the (finite) sequence $\frac{a_k}{\binom{n}{k}}$ is log-concave in k. ∎

Let us briefly recall how to apply this lemma to binomial coefficients and Stirling numbers $c(n, k)$ of the first kind (the number of permutations on the set $[n] = \{1, 2, \ldots, n\}$ with exactly k cycles) and Stirling numbers $S(n, k)$ of the second kind (the number of partitions of $[n]$ into exactly k blocks). The following formulae are well known:

$$(x + 1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \tag{2.1}$$

$$x^{\bar{n}} = \sum_{k=0}^{n} c(n, k) x^k, \tag{2.2}$$

$$x^n = \sum_{k=0}^{n} S(n, k) x^{\underline{n}}, \tag{2.3}$$

where $x^{\underline{k}} = x(x-1)\ldots(x-k+1)$ is the k-th falling power, and $x^{\bar{k}} = x(x+1)\ldots(x+k-1)$ the k-th rising power of x. From (2.1) and (2.2) we see that $(x+1)^n$ and $x^{\bar{n}}$ have only real roots. So, by Newton's lemma we conclude that the sequences $\binom{n}{k}$ and $c(n, k)$ are log-concave. The case of the sequence $S(n, k)$ is a bit more involved. Let

$$P_n(x) = \sum_{k=0}^{n} S(n, k) x^k.$$

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From $P_0(x) = 1$ and from the basic recursion

$$S(n, k) = S(n-1, k-1) + kS(n-1, k),$$

it follows at once that

$$P_n(x) = x[P_n'(x) + P_{n-1}(x)].$$

The function $Q_n(x) = P_n(x)e^x$ has the same roots as $P_n(x)$ and it is easy to verify $Q_n(x) = xQ_n'(x)$. By induction on n and using Rolle's theorem it follows easily that Q_n, and hence P_n, have only real and non-positive roots. So, we conclude:

Theorem 2.2

The sequences $\binom{n}{k}_{k\geq 0}$, $(c(n,k))_{k\geq 0}$ and $(S(n,k))_{k\geq 0}$ are log-concave. Hence, they are unimodal. ∎

An inductive proof of Theorem 2.2 is given in [22].

The next easy lemma is sometimes useful in proving log-convexity results.

Lemma 2.3

Let $f : [a, b] \to \mathbb{R}$ be a positive, continuous function, and

$$I_n = \int_a^b f(x)^n dx, \quad n \geq 1.$$

Then $(I_n)_{n\geq 2}$ is a log-convex sequence.

Proof

By Cauchy-Schwarz inequality, we have

$$I_n^2 = \left(\int_a^b f(x)^n dx \right)^2 = \left(\int_a^b f(x)^{\frac{n-1}{2}} f(x)^{\frac{n+1}{2}} dx \right)^2 \leq \int_a^b f(x)^{n-1} dx \int_a^b f(x)^{n+1} dx = I_{n-1} I_{n+1}.$$

∎

As an example, we apply this lemma to Legendre polynomials $P_n(x)$. It is well known (e.g. [31]) that the following Laplace formula holds:

$$P_n(x) = \frac{1}{\pi} \int_0^\pi (x + \sqrt{x^2 - 1}\cos\varphi)^n d\varphi. \qquad (2.4)$$

Hence, from Lemma 2.3 and (2.4) we obtain

Theorem 2.4

The values $P_n(x)$, $n \geq 0$, for $x \geq 1$ are log-convex. ∎

Another proof of this fact will be presented in Section 4.

We say that a sequence $(a_n)_{n\geq 0}$ has no internal zeros if there do not exist integers $0 \leq i < j < k$ such that $a_i \neq 0$, $a_j = 0$, $a_k \neq 0$.

Theorem 2.5 (Bender-Canfield, see [5])

Let $1, a_1, a_2, \ldots$ be a log-concave sequence of nonnegative real numbers with no internal zeros and let $(b_n)_{n\geq 0}$ be the sequence defined by

$$\sum_{n\geq 0} b_n \frac{x^n}{n!} = exp\left(\sum_{k\geq 1} a_k \frac{x^k}{k!}\right). \qquad (2.5)$$

Then the sequence $(b_n)_{n\geq 0}$ is log-convex and $\left(\frac{b_n}{n!}\right)_{n\geq 0}$ is log-concave. ∎

As our first application of this theorem consider the Bell numbers $(B_n)_{n\geq 0}$. $B_0 = 1$, and B_n is the number of partitions of an n-set. It is well known that the exponential generating function for $(B_n)_{n\geq 0}$ is given by

$$\sum_{n\geq 0} B_n \frac{x^n}{n!} = exp(e^x - 1) = exp\left(\sum_{k\geq 1} \frac{x^k}{k!}\right).$$

By taking $a_k = \frac{1}{(k-1)!}$, $k \geq 1$, and checking that the sequence $1, a_1, a_2, \ldots$ is log-concave, we conclude from Theorem 2.5 that $(B_n)_{n\geq 0}$ is log-convex and $\left(\frac{B_n}{n!}\right)_{n\geq 0}$ is log-concave sequence. More generally,

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for an integer $l \geq 2$, define exp_l to be l times iterated exponential function, i.e.

$$exp_l(x) = exp(exp(\ldots(exp(x))\ldots)),$$

exp written l times. Define the sequence $\left(b_n^{(l)}\right)_{n \geq 0}$ by

$$\sum_{n \geq 0} b_n^{(l)} \frac{x^n}{n!} = exp_l(x),$$

and **Bell numbers of order** l by

$$B_n^{(l)} = \frac{b_n^{(l)}}{exp_l(0)}. \tag{2.6}$$

So, $B_n^{(2)} = B_n$ are ordinary Bell numbers. Now it is not hard to prove by induction on l the following result.

Theorem 2.6

For any fixed $l \geq 2$, the Bell numbers of order l, i.e. the sequence $\left(B_n^{(l)}\right)_{n \geq 0}$ is log-convex, and the sequence $\left(\frac{B_n^{(l)}}{n!}\right)_{n \geq 0}$ is log-concave. ∎

The following lemma is an easy consequence of the definition of log-convexity and log-concavity.

Lemma 2.7

Suppose $(a_n)_{n \geq 0}$ is a positive, log-convex sequence and $a_0 = 1$. Then $a_n a_m \leq a_{n+m}$. If, in addition, $\left(\frac{a_n}{n!}\right)_{n \geq 0}$ is log-concave, then $a_{n+m} \leq \binom{m+n}{n} a_n a_m$. ∎

Now Theorem 2.6 and Lemma 2.7 imply semi-additivity inequalities for Bell numbers of order l.

Corollary 2.8

$$B_n^{(l)} B_m^{(l)} \leq B_{m+n}^{(l)} \leq \binom{m+n}{n} B_n^{(l)} B_m^{(l)}, \quad m, n \geq 0.$$

∎

The next application of the Bender-Canfield theorem concerns the number of certain permutations, i.e. elements $\pi \in \Sigma_n$, the symmetric group of permutations on n letters. For a fixed integer $k \geq 1$ let $c_k(n)$ be the number of all permutations from S_n that have cycles of length not exceeding k. By definition, $c_k(0) = 1$.

Theorem 2.9

The sequence $(c_k(n))_{n \geq 0}$ is log-convex and the sequence $\left(\frac{c_k(n)}{n!}\right)_{n \geq 0}$ is log-concave.

Proof

It is well known (see, e.g. [27]) that the exponential generating function $C_k(x)$ of the sequence $(c_k(n))_{n \geq 0}$ is given by

$$C_k(x) = exp\left(\sum_{j=1}^{k} \frac{x^j}{j}\right).$$

Hence the sequence $(a_k)_{k \geq 0}$ from the Bender-Canfield theorem in (2.5) is as follows: $a_0 = 1, a_1 = 1, \ldots, a_k = 1, a_{k+1} = 0, a_{k+2} = 0, \ldots$, which has no internal zeros and is obviously a log-concave sequence. Hence the claim follows from Theorem 2.6. ∎

The requirement that the sequence (a_n) from Bender-Canfield theorem does not have internal zeros is essential, and in general can not be weakened. As an illustration, let us consider a class of sequences related to $c_k(n)$. Again, for a fixed integer $k \geq 1$ let $e_k(n)$ be the number of all permutations π from Σ_n such that $\pi^k = id$, for $n \geq 1$ and $e_k(0) = 1$. Then the exponential generating function $E_k(x)$ of this sequence is given by (see, e.g. [27])

$$E_k(x) = exp\left(\sum_{j|k} \frac{x^j}{j}\right), \tag{2.7}$$

and the corresponding recurrence is

$$e_k(n) = \sum_{j|k} (n-1)^{\underline{j-1}} e_k(n-j),$$

with appropriate initial conditions. The sequence (a_n) from (2.5) is given by $a_0 = a_1 = 1$, $a_j = 1$ if j divides k and $a_j = 0$ otherwise. This binary sequence $1, 1, a_2, \ldots, a_k, 0, 0, \ldots$ is log-concave if and only if it does not contain $1, 0, 1$ as a subsequence, but it contains internal zeros for all $k > 2$. Taking, for example, $k = 5$, the first few terms of $e_5(n)$ being $1, 1, 1, 1, 1, 25, 145, 505, 1345, \ldots$, we can easily see that this sequence does not exhibit any logarithmically definite behavior, although the sequence $1, 1, 0, 0, 0, 1, 0, \ldots$ is log-concave. For higher values of k, sequences $e_k(n)$ log-behave even more chaotically.

3 The interlacing (or "sandwich") method. Secondary structures.

Let $a(n)$, $n \geq 0$, be a sequence of positive numbers defined by a homogeneous linear recurrence, say of second order:

$$a(n) = R(n)a(n-1) + S(n)a(n-2), \tag{3.1}$$

where R and S are known functions, together with given initial values $a(0) = a_0$, and $a(1) = a_1$.

Let our task be to examine the rate of growth of $a(n)$. We define the sequence of consecutive quotients

$$q(n) = \frac{a(n)}{a(n-1)}, \quad n \geq 1.$$

Dividing (3.1) by $a(n-1)$ we obtain the recurrence

$$q(n) = R(n) + \frac{S(n)}{q(n-1)}, \tag{3.2}$$

with initial condition $q(1) = \frac{a_1}{a_0} := b_1$. The log-concavity or log-convexity of $(a(n))$ is equivalent, respectively, to $q(n) \geq q(n+1)$ or $q(n) \leq q(n+1)$, for all $n \geq 1$. So, what we want to see is whether the sequence $(q(n))_{n \geq 1}$ decreases or increases. To prove that $(q(n))$ increases, it is enough to find an

increasing sequence $(b(n))$ such that

$$b(n) \le q(n) \le b(n+1) \qquad (3.3)$$

holds for all $n \ge 1$, or at least for all $n \ge n_0$ for some n_0. Then we can conclude that $(a(n))$ behaves log-convex at least from some place on. Analogously for log-concavity. This "sandwich method" or "interlacing method" works in some simple cases, but often it is very hard to hit the right sequence $(b(n))$ which is simple enough. In the rest of this section we show how this method works for some combinatorially important sequences. We also show some consequences of the obtained results.

Example 3.1 (Derangements)

Let D_n be the number of derangements on n objects, i.e. the number of permutations $\pi \in \Sigma_n$ without fixed points, for $n \ge 1$, and $D_0 := 1$. It is well known (and easy to prove) that the following recurrence holds:

$$D_n = (n-1)[D_{n-1} + D_{n-2}], \qquad (3.4)$$

with initial conditions $D_0 = 1$, $D_1 = 0$. Then $D_2 = 1$, $D_3 = 2$, $D_4 = 9$, $D_5 = 44$, $D_6 = 265$ etc., and we expect from these initial values that $D_n^2 \le D_{n-1}D_{n+1}$ for $n \ge 3$. Indeed, divide (3.4) by D_{n-1} and denote $q(n) = \frac{D_n}{D_{n-1}}$. Then

$$q(n) = (n-1)\left[1 + \frac{1}{q(n-1)}\right], \qquad (3.5)$$

with $q(3) = 2$, $q(4) = 9/2$. Let $b(n) = n - 1/2$. It is easy to check by induction on n and using (3.5) that

$$b(n) \le q(n) \le b(n+1),$$

for all $n \ge 4$. Since $(b(n))$ is clearly increasing and $q(3) \le q(4)$, we conclude that (D_n) is log-convex for $n \ge 3$. ∎

Example 3.2

Let $T_2(n)$ denotes the number of $n \times n$ symmetric $\mathbb{N}_0$-matrices with every row (and hence every column) sum equal to 2 with trace zero (i.e. all main-diagonal entries are zero) (Example 5.2.8 in [27]). The exponential generating function of $T_2(n)$ is given by

$$T(x) = \frac{1}{\sqrt{1-x}} exp\left(\frac{x^2}{4} - \frac{x}{2}\right).$$

The numbers $T_2(n)$ satisfy the recurrence

$$T_2(n) = (n-1)T_2(n-1) + (n-1)T_2(n-2) - \binom{n-1}{2}T_2(n-3)$$

with the initial conditions $T_2(0) = 1$, $T_2(1) = 0$, $T_2(2) = T_2(3) = 1$. The corresponding recurrence for successive quotients $q_2(n) = T_2(n)/T_2(n-1)$ is given by

$$q_2(n) = (n-1) + \frac{n-1}{q_2(n-1)} - \binom{n-1}{2}\frac{1}{q_2(n-1)q_2(n-2)}, \quad n \geq 5,$$

with the initial conditions $q_2(3) = 1$, $q_2(4) = 6$. Tabulating the first few values of $q_2(n)$, we see that, after some initial fluctuations, this sequence seems to behave like $n - \frac{1}{2}$. So, we guess that $n - 1 \leq q_2(n) \leq n$, and indeed, this follows easily by induction on n for $n \geq 6$. Hence, the sequence $(T_2(n))_{n \geq 6}$ is log-convex. ∎

For our next application we need some preparations. A **Motzkin path** of length n is a lattice path in (x, y)-plane from $(0, 0)$ to $(n, 0)$ with steps $(1, 1)$ (or Up), $(1, -1)$ (or $Down$) and $(1, 0)$ (or $Level$), never falling below the x-axis. Denote by $\mathcal{M}(n)$ the set of all Motzkin paths of length n. The number $M_n = |\mathcal{M}(n)|$ is the n-th **Motzkin number**. By definition, $M_0 = 1$.

A handful of other combinatorial interpretations of M_n are listed in Ex. 6.38 in [27]. A typical member of the Motzkin family $\mathcal{M}(20)$ is shown in Fig. 1 as a "landscape path". A **peak** of a Motzkin path is a place where an Up step is immediately followed by a $Down$ step. A **plateau** of length l is a
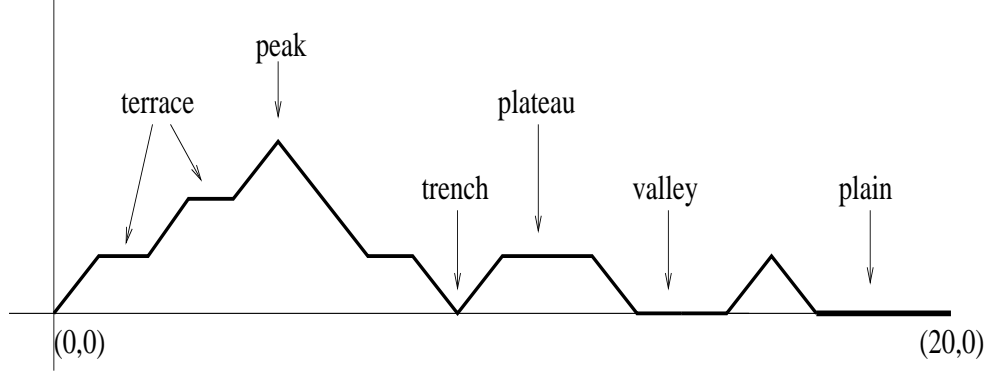
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Figure 1: A Motzkin path of length 20

sequence of l consecutive *Level* steps, immediately preceded by an *Up* step and immediately followed by a *Down* step. Similarly we define a **terrace**, **trench**, **valley** and **plain** of a Motzkin path.

A Motzkin path without any *Level* steps is called a **Dyck path** or a "mountain path". It is well known that the set $\mathcal{D}(n)$ of all Dyck paths of length $2n$ is enumerated by **Catalan numbers C_n**, i.e. $|\mathcal{D}(n)| = C_n = \frac{1}{n+1}\binom{2n}{n}$. From the explicit formula it follows immediately that $C_n^2 \leq C_{n-1}C_{n+1}$, i.e. Catalan numbers are log-convex. It is also easy to find a simple combinatorial proof of this fact. By counting the number of *Level* steps on a Motzkin path, it can be easily shown that Catalan and Motzkin numbers are related as follows:

$$M_n = \sum_{k \geq 0} \binom{n}{2k} C_k, \quad C_{n+1} = \sum_{k \geq 0} \binom{n}{k} M_k.$$

Proposition 3.2

a) The Motzkin numbers satisfy the following convolutive recursion:

$$M_{n+1} = M_n + \sum_{k=0}^{n-1} M_k M_{n-k-1}. \tag{3.6}$$

b) The generating function of (M_n) is given by

$$M(x) = \sum_{n \geq 0} M_n x^n = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}. \tag{3.7}$$

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c) M_n's satisfy the short recursion

$$(n+2)M_n = (2n+1)M_{n-1} + 3(n-1)M_{n-2}. \qquad (3.8)$$

Proof

a) $M_0 = 1$ and $M_1 = 1$. A Motzkin path of length $n+1$ either starts by a *Level* step and then can proceed in M_n ways, or starts by an *Up* step and returns for the first time to the x-axis after $k+1$ steps (to the point $(k+2, 0)$). The number of latter is equal to the number of pairs (P_1, P_2), where P_1 is a (translated) Motzkin path from $(1,1)$ to $(k+1,1)$ which is not below the line $y = 1$, and P_2 is a (translated) Motzkin path from $(k+2,0)$ to $(n+1,0)$. The number of paths P_1 is equal to M_k, while the number of paths P_2 is equal to the number of Motzkin paths on $n+1-(k+2) = n-k-1$ steps, and this is M_{n-k-1}. Thus a) follows.

b) If we multiply (3.6) by x^{n+1} and sum over $n \geq 0$, we get the functional equation $x^2 M^2(x) + (x - 1)M(x) + 1 = 0$, and since $M(0) = M_0 = 1$, we obtain (3.7).

c) The generating function $M(x)$ is algebraic, hence D-finite. Therefore (M_n) is P-recursive. Now from Eq(6.38) in [27] for the polynomial $A(x) = 1 - 2x - 3x^2$ and $r = 2$, $a_0 = 1$, $a_1 = -2$, $a_2 = -3$, $d = 2$, we get the claim. ∎

Theorem 3.3

a) The sequence M_n of Motzkin numbers is log-convex.

b) $M_n/M_{n-1} < 3$, for all $n \geq 1$.

c) The sequence M_n/M_{n-1} is convergent and $\lim_{n\to\infty} M_n/M_{n-1} = 3$.

Proof

a) Divide (3.8) by $(n+2)M_{n-1}$ and let $q(n) := M_n/M_{n-1}$. Then we obtain

$$q(n) = \frac{1}{n+2}\left[2n+1+\frac{3(n-1)}{q(n-1)}\right], \quad n \geq 2, \tag{3.9}$$

with the initial condition $q(1) = 1$. Then $q(2) = 2$. Now define the sequence $b_n = 3\frac{2n}{2n+3}$. This

sequence is obviously increasing. We claim that $b_n \leq q(n) \leq b_{n+1}$, for all $n \geq 3$. This is obviously

true for $n = 3$. Let $n \geq 3$. By inductive hypothesis, we have

$$q(n) = \frac{2n+1}{n+2} + \frac{3(n-1)}{n+2}\frac{1}{q(n-1)} \geq \frac{2n+1}{n+2} + \frac{3(n-1)}{n+2}\frac{1}{b_n}.$$

But

$$\frac{2n+1}{n+2} + \frac{3(n-1)}{n+2}\frac{1}{b_n} - b_n = \frac{3}{2}\frac{n-3}{n(n+2)(2n+3)} \geq 0,$$

for all $n \geq 3$, and the inequality $q(n) \geq b_n$ follows. On the other hand, by the inductive hypothesis

$b_{n-1} \leq q(n-1)$, we have

$$q(n) \leq \frac{2n+1}{n+2} + \frac{3(n-1)}{n+2}\frac{1}{b_{n-1}}.$$

Subtracting b_{n+1} from the right hand side, we get

$$\frac{2n+1}{n+2} + \frac{3(n-1)}{n+2}\frac{1}{b_{n-1}} - b_{n+1} = -\frac{1}{2}\frac{1}{(n+2)(2n+5)} \leq 0,$$

and the inequality $q(n) \leq b_{n+1}$ follows. So, the claim is proved by induction. Hence, the sequence

M_n/M_{n-1} is increasing, and the Motzkin sequence is log-convex.

b) We have proved in a) that $M_n/M_{n-1} \leq b_n = 3\frac{2n}{2n+3} < 3$.

c) By a) and b) it follows that $q(n) = M_n/M_{n-1}$ is an increasing and bounded sequence: $2 \leq q(n) < 3$,

hence convergent. Passing to limit in (3.9), or passing to limit in the sandwich inequality $b_n \leq q(n) \leq$

b_{n+1} above, the last claim in c) follows. ∎

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Corollary 3.4

a) The sequence $(M_n/n!)$ is log-concave.

b) $M_n^2 \leq M_{n-1}M_{n+1} \leq \left(1 + \frac{1}{n}\right) M_n^2$, for all $n \geq 1$.

c) $M_m M_n \leq M_{m+n} \leq \binom{m+n}{n} M_m M_n$, for all $m, n \geq 0$.

Proof

a) The log-concavity of $M_n/n!$ is equivalent to $\frac{M_{n+1}}{M_n} \leq \frac{n+1}{n} \frac{M_n}{M_{n-1}}$. So, we have to prove that $q(n+1) \leq \frac{n+1}{n}q(n)$. We know that the sequence $q(n)$ is increasing. Starting from the short recursion (3.9) for $q(n+1)$, we have

$$
\begin{aligned}
q(n+1) &= \frac{2n+3}{n+3} + \frac{3n}{n+3}\frac{1}{q(n)} \leq \frac{2n+3}{n+3} + \frac{3n}{n+3}\frac{1}{q(n-1)} \\
&= \frac{n+2}{n+3}\frac{n}{n-1}\left[\frac{(2n+3)(n-1)}{2n(n+1)}\frac{2n+1}{n+2} + \frac{3(n-1)}{n+2}\frac{1}{q(n-1)}\right].
\end{aligned}
$$

The claim now follows by noting that the term $\frac{(2n+3)(n-1)}{2n(n+1)}$ is clearly less than one for all $n \geq 1$, and that the inequality $\frac{n+2}{n+3}\frac{n}{n-1} < \frac{n+1}{n}$ is valid for all $n \geq 3$. The validity of our claim can easily be checked for $1 \leq n \leq 3$.

b) This follows from log-convexity of Motzkin numbers and a).

c) A simple combinatorial proof of the left inequality follows from the fact that a concatenation of two Motzkin paths of lengths m and n, respectively, is again a valid Motzkin path of length $m + n$. To prove the right inequality, start from $q(n) \geq \frac{n}{n+1}q(n+1)$. Using this inequality repeatedly, we get

$$
\frac{M_1}{M_0} \geq \frac{1}{2}\frac{M_2}{M_1} \geq \frac{1}{3}\frac{M_3}{M_2} \geq \ldots \geq \frac{1}{m+n}\frac{M_{m+n}}{M_{m+n-1}},
$$

for all $n \geq 0$, $m \geq 1$.

Hence, for any $0 \leq j \leq m - 1$, we have

$$\frac{M_{j+1}}{M_j} \geq \frac{j+1}{m+n} \frac{M_{m+n}}{M_{m+n-1}}.$$

From this we get

$$\frac{M_1}{M_0} \frac{M_2}{M_1} \cdots \frac{M_m}{M_{m-1}} \geq \left(\frac{1}{n+1} \frac{M_{n+1}}{M_n} \right) \left(\frac{2}{n+2} \frac{M_{n+2}}{M_{n+1}} \right) \cdots \left(\frac{m}{m+n} \frac{M_{m+n}}{M_{m+n-1}} \right),$$

and after the cancellation:

$$\frac{M_m}{M_0} \geq \frac{m!n!}{(m+n)!} \frac{M_{m+n}}{M_n}.$$

Since $M_0 = 1$, we get the claim. The case $m = 0$, $n \geq 0$ is trivial. ∎

Remark 3.5

An algebraic proof of log-convexity of Motzkin numbers was given in [1], and a combinatorial proof in [8]. We shall give yet another ("calculus") proof later in Section 4.

Now we proceed in applying the "sandwich method" to combinatorial structures that generalize both Dyck and Motzkin structures in a sense that they are counted by Catalan and Motzkin numbers in some special cases. They are called secondary structures and come from molecular biology (see, e.g. [29], [33], [34], [17], [23], [19]). More details are in [12] and [11].

Let n and l are integers, $n \geq 1$, $l \geq 0$. A **secondary structure** of **size** n and **rank** l is a labeled graph S on the vertex set $V(S) = [n] = \{1, 2, \dots, n\}$ whose edge set $E(S)$ consists of two disjoint subsets, $P(S)$ and $H(S)$, satisfying the following conditions:

(a) $\{i, i+1\} \in P(S)$, for all $1 \leq i \leq n - 1$;

(b) $\{i, j\} \in H(S)$ and $\{i, k\} \in H(S) \implies j = k$;

(c) $\{i, j\} \in H(S) \implies |i - j| > l$;

(d) $\{i, j\} \in H(S), \{p, q\} \in H(S)$ and $i < p < j \implies i < q < j$.

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The vertices of a secondary structure are (in biology) usually called **bases**, the edges in $P(S)$ are called *p*-**bonds** and those in $H(S)$ *h*-**bonds**. A secondary structure S with $H(S) = \emptyset$ is called **trivial**. The number of *h*-bonds of S is called its **order**. An example of a secondary structure of size 12, rank 2 and order 3 is shown below in Fig. 2. Note that every *h*-bond "leaps" over at least two bases. We
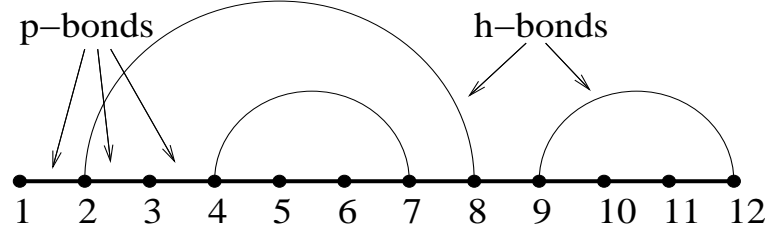

Figure 2: An example of a secondary structure

denote the set of all secondary structures of size n and rank l by $\mathcal{S}^{(l)}(n)$, and the set of such structures of order k by $\mathcal{S}_k^{(l)}(n)$. The respective cardinalities we denote by $S^{(l)}(n)$ and $S_k^{(l)}(n)$. By definition, we put $S^{(l)}(0) = 1$, for all l.

Another interpretation of secondary structures is as follows. Denote by $\mathcal{M}^{(l)}(n)$ the set of all Motzkin paths in $\mathcal{M}(n)$ whose every plateau is at least l steps long. For $l = 0$, $\mathcal{M}^{(0)}(n) = \mathcal{M}(n)$, i.e. all Motzkin paths on n steps.

Proposition 3.6

There is a bijection between $\mathcal{S}^{(l)}(n)$ and $\mathcal{M}^{(l)}(n)$ for all $n \geq 1$, $l \geq 0$.

Proof

We shall only briefly describe the correspondence $\mathcal{S}^{(l)}(n) \rightarrow \mathcal{M}^{(l)}(n)$. Take a secondary structure $S \in \mathcal{S}^{(l)}(n)$ and scan it from left to right. To each base, starting from 1, we assign a step in a lattice path as follows. To any unpaired base (i.e. base in which no *h*-bond starts or ends) we assign a *Level* step. To a base in which a new *h*-bond starts assign an *Up* step, and to each base in which an already

encountered h-bond terminates, assign a *Down* step. The obtained path P is in $\mathcal{M}^{(l)}(n)$ and it is not hard to check that $S \longmapsto P$ is a bijection. ∎

An example of the correspondence $\mathcal{S}^{(1)}(10) \longleftrightarrow \mathcal{M}^{(1)}(10)$ is shown in Fig. 3. Therefore, the rank 0
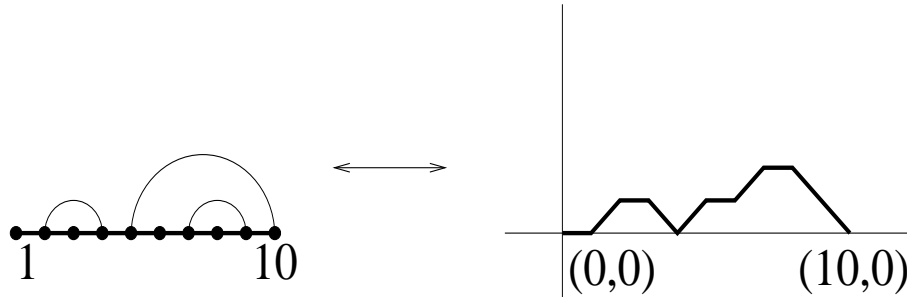

Figure 3: A secondary structure and the corresponding Motzkin path

secondary structures, i.e the border case, are just all Motzkin paths, hence $S^{(0)}(n) = M_n$. Note that for $l \geq 1$ a secondary structure of rank l is a simple graph.

If we allow rank to degenerate to the value of $l = -1$, then an h-bond can terminate in the very vertex it starts from, i.e. we allow loops.

Proposition 3.7

There is a bijection between the set $\mathcal{S}^{(-1)}(n)$ of all secondary structures of size n and rank -1 and the set $\mathcal{D}(n+1)$ of all Dyck paths on $2(n+1)$ steps. Hence, $S^{(-1)}(n) = C_{n+1}$, for all $n \geq 0$.

Proof

Again, we shall only briefly describe how to assign a member of $\mathcal{S}^{(-1)}(n)$ to a Dyck path $P \in \mathcal{D}(n+1)$. So, take a Dyck path on $2(n+1)$ steps. Discard the first and the last step and divide the remaining path in groups of two consecutive steps. Assign to each group a base in a secondary structure according to the following rule. To a group of two Up steps assign a base in which an h-bond starts, to a group

of two *Down* steps assign a base in which the last started h-bond terminates. To a group $(Up, Down)$

assign a base with a loop attached to it, and, finally, to a group $(Down, Up)$ assign an unpaired base.

The obtained graph S is in $\mathcal{S}^{(-1)}(n)$ and $P \longmapsto S$ is a bijection. ■

An example of the correspondence $\mathcal{D}(8) \longleftrightarrow \mathcal{S}^{(-1)}(7)$ is shown in Fig. 4. The third combinatorial
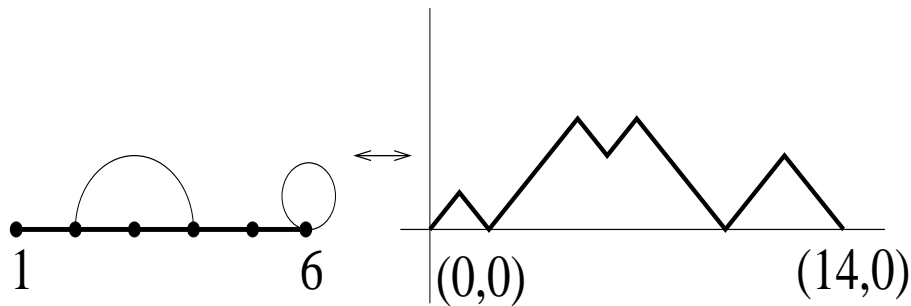

Figure 4: The correspondence between secondary structures of rank -1 and Dyck paths

interpretation of secondary structures is related to pattern avoidance in permutations. A **pattern** is

a permutation $\sigma \in \Sigma_k$, and a permutation $\pi \in \Sigma_n$ **avoids** σ if there is no subsequence in π whose

members are in the same relative order as the members of σ. It is well known ([18]) that the number

of permutations from Σ_n avoiding $\sigma \in \Sigma_3$ is equal to the Catalan number C_n, for all patterns $\sigma \in \Sigma_3$.

The concept of pattern avoidance was generalized in [3], by allowing the requirement that two letters

adjacent in a pattern must be adjacent in the permutation. An example of a generalized pattern is

$1 - 32$, where an $1 - 32$ sub-word of a permutation $\pi = a_1 a_2 \ldots a_n$ is any sub-word $a_i a_j a_{j+1}$ such that

$i < j$ and $a_i < a_{j+1} < a_j$. Generalized pattern avoidance is treated in more detail in [9], where it

is shown that the permutations from Σ_n that avoid both $1 - 23$ and $13 - 2$ are enumerated by the

Motzkin numbers.

Proposition 3.8

For any $l \geq 0$, there is a bijection between $\mathcal{S}^{(l)}(n)$ and the set of all permutations from Σ_n that avoid

$\{1 - 23, 13 - 2, ij\}$, where $j \geq i + l$.

Proof

This bijection is just the bijection given in the proof of Proposition 24 in [9]. It is easy to see that any pattern $i, i + k$ in a permutation $\sigma \in \Sigma_n$ avoiding $\{1 - 23, 13 - 2\}$ generates a plateau of length $k - 1$ in the corresponding Motzkin path. So, the claim follows from Proposition 3.6. ∎

Let us find now the recurrences and generating function for the general secondary structure numbers analogous to those for the Motzkin numbers (the border case) as was shown in Proposition 3.2.

Proposition 3.9

a) For any fixed integer $l \geq -1$, the numbers $S^{(l)}(n)$ of secondary structures of rank l satisfy the following convolutive recurrence

$$S^{(l)}(n + 1) = S^{(l)}(n) + \sum_{m=l}^{n-1} S^{(l)}(m)S^{(l)}(n - m - 1), \quad n \geq l + 1, \tag{3.10}$$

together with the initial conditions

$$S^{(l)}(0) = S^{(l)}(1) = \ldots = S^{(l)}(l + 1) = 1. \tag{3.11}$$

b) The generating function $S^{(l)}(x) = \sum_{n \geq 0} S^{(l)}(n)x^n$ is given by

$$S_l(x) = \frac{-\omega_l(x) - \sqrt{\omega_l^2(x) - 4x^2}}{2x^2}, \tag{3.12}$$

where

$$\omega_l(x) = x - 1 - x^2 - \ldots - x^{l+1} = 2x - (1 + x + x^2 + \ldots + x^{l+1}) = 1 - x^2 \frac{1 - x^l}{1 - x}. \tag{3.13}$$

c) The sequence $\left(S^{(l)}(n)\right)_{n \geq 0}$ satisfies the following "short" recursion

$$(n + 2)S^{(l)}(n) = \sum_{k=1}^{2l+2} A^{(l)}(n, k)S^{(l)}(n - k), \tag{3.14}$$

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where

$$A^{(l)}(n,k) = \begin{cases} -\frac{1}{2}(k-3)(2n+4-3k) & , & 1 \le k \le l+1 \\[2mm] -\frac{1}{2}(l-3)(2n-3l-2) & , & k = l+2 \\[2mm] -\frac{1}{2}(2l+3-k)(2n+4-3k) & , & l+3 \le k \le 2l+2. \end{cases}$$

Proof

a) Clearly, for $n \le l+1$ there is only one (the trivial) secondary structure of rank l and size n, and hence initial conditions (3.11) hold. Let $n \ge l+1$. A secondary structure on $n+1$ bases either does not contain an h-bond starting at the base 1, in which case there are $S^{(l)}(n)$ such structures, or it has an h-bond from the base 1 to some base $m+2$, at least l bases apart. In this case, there are $S^{(l)}(m)S^{(l)}(n-m-1)$ such structures, and (3.10) follows.

b) Multiplying (3.10) by x^{n+1}, summing over $n \ge 0$ and taking into account the initial conditions (3.11), we obtain the functional equation

$$x^2[S_l(x)]^2 + \omega_l(x)S_l(x) + 1 = 0,$$

and this, in turn, implies (3.12).

c) As in the proof of Proposition 3.2 c), we use D-finiteness of the above generating function and again formula (6.38) in [27] with $d = 2$, $r = 2l+2$ to obtain the claim. More details are in [11] and [12]. ■

Let us write down explicitly the recurrences (3.14) in cases $l = 1, 2, 3$ (of course, for $l = 0$, it coincides with (3.8)):

$$S^{(1)}(n) = \frac{2n+1}{n+2}S^{(1)}(n-1) + \frac{n-1}{n+2}S^{(1)}(n-2) + \frac{2n-5}{n+2}S^{(1)}(n-3) - \frac{n-4}{n+2}S^{(1)}(n-4), \quad (3.15)$$

$$S^{(1)}(0) = S^{(1)}(1) = S^{(1)}(2) = 1, S^{(1)}(3) = 2;$$

$$S^{(2)}(n) = \frac{2n+1}{n+2}S^{(2)}(n-1) + \frac{n-1}{n+2}S^{(2)}(n-2) + \frac{n-4}{n+2}S^{(2)}(n-4) - \frac{2n-11}{n+2}S^{(2)}(n-5) - \frac{n-7}{n+2}S^{(2)}(n-6),$$

$$\tag{3.16}$$

$$S^{(2)}(0) = S^{(2)}(1) = S^{(2)}(2) = S^{(2)}(3) = 1, S^{(2)}(4) = 2, S^{(2)}(5) = 4;$$

$$S^{(3)}(n) = \frac{2n+1}{n+2}S^{(3)}(n-1) + \frac{n-1}{n+2}S^{(3)}(n-2) - \frac{n-4}{n+2}S^{(3)}(n-4)$$

$$-\frac{3n-21}{n+2}S^{(3)}(n-6) - \frac{2n-17}{n+2}S^{(3)}(n-7) - \frac{n-10}{n+2}S^{(3)}(n-8), \tag{3.17}$$

$$S^{(3)}(0) = S^{(3)}(1) = S^{(3)}(2) = S^{(3)}(3) = S^{(3)}(4) = 1, S^{(3)}(5) = 2, S^{(3)}(6) = 4, S^{(3)}(7) = 8.$$

Remark 3.10

The recurrences (3.14) do not have any polynomial solutions or solutions in hypergeometric terms for

$l \geq 0$. This fact follows by applying the algorithm **Hyper**, described in [21].

Finally, we now return to our main theme: the log-convexity of secondary structure numbers.

Theorem 3.11

The sequence $\left(S^{(1)}(n)\right)_{n\geq 0}$ is log-convex.

Proof

We start from the short recursion (3.15). Dividing (3.15) by $S^{(1)}(n-1)$ and denoting $S^{(1)}(n)/S^{(1)}(n-1)$

by q_n, we obtain the following recursion for the numbers q_n:

$$q_n = \frac{1}{n+2}\left[2n+1+\frac{n-1}{q_{n-1}}+\frac{2n-5}{q_{n-1}q_{n-2}}-\frac{n-4}{q_{n-1}q_{n-2}q_{n-3}}\right], \quad n \geq 4, \tag{3.18}$$

with the initial conditions $q_1 = q_2 = 1$, $q_3 = 2$. It is easy to check that $q_1 \leq q_2 \leq \ldots \leq q_6$.

Assume, for the moment (and we shall prove it later on) that the sequence (q_n) is convergent with limit q when $n \to \infty$. By passing to limit in (3.18), we obtain the equation $q^4 - 2q^3 - q^2 - 2q + 1 = 0$, whose maximal positive solution is $q = \varphi^2 = \varphi + 1$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Define now the sequence $a_n = \frac{2n}{2n+3}\varphi^2$. It is clearly an increasing sequence and its limit is φ^2. We claim that (a_n) is interlaced with our sequence (q_n). More precisely, we shall prove by induction that

$$a_n \le q_n \le a_{n+1}, \tag{3.19}$$

for $n \ge 6$.

First we check directly the cases $n = 6, 7, 8$ and 9. Now take $n \ge 9$. From the induction hypothesis and (3.18), we have

$$
\begin{aligned}
(n+2)q_n &= 2n + 1 + \frac{n-1}{q_{n-1}} + \frac{n-1}{q_{n-1}q_{n-2}} + \frac{n-4}{q_{n-1}q_{n-2}} - \frac{n-4}{q_{n-1}q_{n-2}q_{n-3}} \\
&= 2n + 1 + \frac{n-1}{q_{n-1}} + \frac{n-1}{q_{n-1}q_{n-2}} + (n-4)\frac{q_{n-3}-1}{q_{n-1}q_{n-2}q_{n-3}} \\
&\ge 2n + 1 + \frac{n-1}{a_n} + \frac{n-1}{a_n a_{n-1}} + (n-4)\frac{a_{n-3}-1}{a_n a_{n-1} a_{n-2}}.
\end{aligned}
$$

We would like the right hand side to be at least $(n+2)a_n$. But this is equivalent to

$$(2n+1)a_n a_{n-1} a_{n-2} + (n-1)a_{n-1}(a_{n-2}+1) + (n-4)(a_{n-3}+1) - (n+2)a_n^2 a_{n-1} a_{n-2} \ge 0.$$

Inserting the formulae for a_n's, we get

$$\frac{12(5+\sqrt{5})n^4 - 2(241+121\sqrt{5})n^3 + 2(847+382\sqrt{5})n^2 - 3(341+146\sqrt{5})n + 126 + 54\sqrt{5}}{(2n-3)(2n-1)(2n+1)(2n+3)^2} \ge 0.$$

The denominator is positive for all integers $n \ge 2$. Denote the numerator by $L(n)$ and shift its argument for 6. The polynomial $L(n+6)$ has only positive coefficients, so it can not have a positive root. It then follows that $L(n)$ can not have a root $\gamma \ge 6$. So the left inequality is valid for all $n \ge 6$, and hence $q_n \ge a_n$.

To prove the other inequality, note that the induction hypothesis implies

$$(n+2)q_n \leq 2n + 1 + \frac{n-1}{a_{n-1}} + \frac{n-1}{a_{n-1}} + \frac{n-1}{a_{n-1}a_{n-2}} + (n-4)\frac{a_{n-2}-1}{a_{n-1}a_{n-2}a_{n-3}}.$$

The condition that the right hand side of this inequality does not exceed $(n+2)a_{n+1}$ is equivalent to

$$(2n+1)a_{n-1}a_{n-2}a_{n-3} + (n-1)a_{n-3}(a_{n-2}+1) + (n-4)(a_{n-2}-1) - (n+2)a_{n+1}a_{n-1}a_{n-2}a_{n-3} \leq 0.$$

Substituting the formulae for a_n's, we get

$$-3\frac{(82+42\sqrt{5})n^3 - (572+248\sqrt{5})n^2 + (1103+474\sqrt{5})n - (529+247\sqrt{5})}{(2n-3)(2n-1)(2n+1)(2n+5)} \leq 0$$

If we put $n+5$ instead of n in the numerator, we get a polynomial with all the coefficients positive, and from this we conclude that the numerator does not change the sign for $n \geq 6$. So, we have proved the inequality $q_n \leq a_{n+1}$, and thus completed the induction step. This proves the theorem. ∎

Corollary 3.12

The sequence $q_n = \frac{S^{(1)}(n)}{S^{(1)}(n-1)}$ is strictly increasing for all $n \geq 5$, bounded from above by φ^2 and $\lim_{n\to\infty} q_n = \varphi^2 = \frac{3+\sqrt{5}}{2}$. ∎

Remark 3.13

It is proved in [12] that the asymptotic behavior of $S^{(l)}(n)$, $n \geq 0$, is given by $S^{(l)}(n) \sim K_l \alpha_l^n n^{-3/2}$, where K_l and α_l are constants depending only on the rank l. Denote $q_n^{(l)} = S^{(l)}(n)/S^{(l)}(n-1)$. Then it follows

$$q_n^{(l)} \sim \alpha_l \left(1 - \frac{1}{n}\right)^{3/2} \nearrow \alpha_l.$$

In other words $q_n^{(l)}$ asymptotically behaves as an increasing sequence tending to α_l as $n \to \infty$. It can be shown that $\alpha_l \in [2,3]$. The exact values of α_l are known for $l \leq 6$. So, $\alpha_0 = 3$, $\alpha_1 = (3+\sqrt{5})/2$, $\alpha_2 = 1 + \sqrt{2}$, etc., and $\alpha_l \searrow 2$ as $l \to \infty$.

Theorem 3.14

The sequences $S^{(l)}(n)$ are log-convex, for $l = 2, 3$ and 4. ∎

Outline of the proof

We present only the case $l = 2$. Dividing the short recursion for $S^{(2)}(n)$ by $S^{(2)}(n-1)$ and denoting the quotient $\frac{S^{(2)}(n)}{S^{(2)}(n-1)}$ by $q_n^{(2)}$, we obtain the recursion for the sequence $(q_n^{(2)})$

$$q_n^{(2)} = \frac{1}{n+2}\left[2n + 1 + \frac{n-1}{q_{n-1}^{(2)}} + \frac{n-4}{q_{n-1}^{(2)}q_{n-2}^{(2)}q_{n-3}^{(2)}} - \frac{2n-11}{q_{n-1}^{(2)}q_{n-2}^{(2)}q_{n-3}^{(2)}q_{n-4}^{(2)}} - \frac{n-7}{q_{n-1}^{(2)}q_{n-2}^{(2)}q_{n-3}^{(2)}q_{n-4}^{(2)}q_{n-5}^{(2)}}\right]$$

with the initial conditions $q_1^{(2)} = q_2^{(2)} = q_3^{(2)} = 1$, $q_4^{(2)} = q_5^{(2)} = 2$. We want to prove that the sequence $(q_n^{(2)})$ is increasing.

From Remark 3.13 we conclude that the sequence $(q_n^{(2)})$ behaves asymptotically as $(1+\sqrt{2})(1-\frac{1}{n})^{3/2}$. Denote this quantity by b_n, i.e. $b_n = (1+\sqrt{2})(1-\frac{1}{n})^{3/2}$. Now take the first three terms of the series expansion of b_n in powers of $\frac{1}{n+2}$. Define

$$a_n = \alpha_2\left(1 - \frac{3}{2}\frac{1}{n+2} + \frac{3}{8}\frac{1}{(n+2)^2}\right) = \alpha_2\frac{8n^2 + 20n + 11}{8(n+2)^2},$$

where $\alpha_2 = 1 + \sqrt{2}$. The sequence (a_n) tends increasingly toward α_2. We shall show now that the sequences $(q_n^{(2)})$ and (a_n) are interlaced, i.e.

$$a_{n-1} \leq q_n^{(2)} \leq a_n,$$

for sufficiently large n.

Suppose inductively that $a_{n-i-1} \leq q_{n-1}^{(2)} \leq a_{n-i}$ for $i = 1, 2, 3, 4, 5$. Then

$$q_n^{(2)} \leq \frac{1}{n+2}\left[2n + 1 + \frac{n-1}{a_{n-2}} + \frac{n-4}{a_{n-2}a_{n-3}a_{n-4}} - \frac{2n-11}{a_{n-1}a_{n-2}a_{n-3}a_{n-4}} - \frac{n-7}{a_{n-1}a_{n-2}a_{n-3}a_{n-4}a_{n-5}}\right].$$

If we prove that the right hand side of this inequality does not exceed a_n, the right inequality, $q_n^{(2)} \leq a_n$,

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will follow. But this is equivalent to the condition

$$\frac{P_{10}(n)}{Q_{12}(n)} \geq 0,$$

where $P_{10}(n)$ and $Q_{12}(n)$ are certain polynomials in n of degree 10 and 12, respectively. (Using *Mathematica*, the polynomials $P_{10}(n)$ and $Q_{12}(n)$ can easily be computed explicitly.) Their leading coefficients are $262144(1+\sqrt{2})$ and $8^6(1+\sqrt{2})^5$, respectively, and we can conclude that this quotient is positive for all n big enough. Again, the biggest real roots of the polynomials $P_{10}(n)$ and $Q_{12}(n)$ can be easily found using *Mathematica*. It turns out that their quotient becomes (and remains) positive for $n \geq 39$.

On the other hand, from the induction hypothesis and the recursion for $q_n^{(2)}$ it follows that

$$q_n^{(2)} \geq \frac{1}{n+2}\left[2n+1+\frac{n-1}{a_{n-1}}+\frac{n-4}{a_{n-1}a_{n-2}a_{n-3}}-\frac{2n-11}{a_{n-2}a_{n-3}a_{n-4}a_{n-5}}-\frac{n-7}{a_{n-2}a_{n-3}a_{n-4}a_{n-5}a_{n6}}\right].$$

That the right hand side of this inequality is $\geq a_{n-1}$ is equivalent to $\frac{P_{13}(n)}{Q_{15}(n)} \geq 0$, where P_{13} and Q_{15} are certain polynomials in n of degree 13 and 15, respectively, with the positive leading coefficients. Their quotient is positive for n big enough ($n \geq 6$).

The claim now follows by checking that $(q^{(2)}(n))$ is increasing for $n \leq 44$.

A similar proof works for $l = 3$ and $l = 4$. We omit the details. ∎

4 Calculus method

Let us start again as in (3.1) with a linear homogeneous recursion for positive numbers $a(n)$, and consider again the corresponding recurrence (3.2) for the quotients $q(n) = a(n)/a(n-1)$. Suppose

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again we want to prove that $a(n)$ is log-convex, i.e. that $(q(n))$ is an increasing sequence (at least from some place n_0 on).

This time we do the following. Define a continuous function $f : [1, \infty) \to \mathbb{R}$ (or $f : [n_0, \infty) \to \mathbb{R}$) starting with the appropriate linear function on the segment $[1, 2]$ (or $[n_0, n_0 + 1]$, for some $n_0 \in \mathbb{N}$) determined by the initial and the next value of $q(n)$, and then continue to the next segment by the same rule as (3.2). In other words, by replacing $q \to f$, $n \to x$, (3.2) becomes

$$f(x) = R(x) + \frac{S(x)}{f(x-1)}, \tag{4.1}$$

defined so for $x \geq 2$ (or $x \geq n_0 + 1$). In a sense, f is a dynamical system patching the discrete values $f(n) = q(n)$ if R and S are "good" enough functions. For example, if R and S are rational functions (and in combinatorics this is mostly the case) without poles on the positive axis, then f is a piecewise rational function, i.e. rational on every open interval $(n, n+1)$ for any integer $n \geq 1$ (or $n \geq n_0$). If we can prove that f is smooth on such open intervals (usually by proving that f is bounded from above and below by some well-behaved functions), we can consider the derivative $f'(x)$, for any $x \in (n, n+1)$. The idea is to show inductively on n that $f'(x) \geq 0$ (or $f'(x) \leq 0$ if we want to prove log-concavity of $a(n)$). This will imply that f is an increasing function (or a decreasing function in the log-concave case) on any open interval $(n, n+1)$, and then, by continuity of f it will follow that f is increasing (or decreasing) on its whole domain, and hence $q(n) = f(n)$, $n \in \mathbb{N}$, will increase (or decrease), too.

Now, in general, if we want to prove that a sequence $q(n)$ defined by (3.2) is increasing, we form the corresponding functional equation (4.1) with the appropriate start, i.e. we define $f(x)$ on some starting segment $[n_0, n_0 + 1]$ to be an increasing function and then prove inductively on n that $f'(x) \geq 0$ for $x \in (n, n+1)$. It is always necessary to have some a priori bounds for f. So, suppose we know $0 < m(x) \leq f(x) \leq M(x)$, for all $x \geq n_0$. Let us find some sufficient conditions which ensure that $f'(x) \geq 0$. Of course, we assume that R and S are smooth on all open intervals $(n, n+1)$, $n \geq n_0$.

Fix an $x \in (n, n+1)$ and write $f = f(x)$, $f_1 = f(x-1)$, $R = R(x)$, $S = S(x)$. Then (4.1) can be written as

$$f = R + \frac{S}{f_1},$$

or equivalently

$$f f_1 = R f_1 + S. \tag{4.2}$$

Taking the derivative d/dx of both sides of (4.2), we get

$$f' f_1 + f f_1' = R' f_1 + R f_1' + S',$$

implying

$$f' f_1 = R' f_1 + S' + (R - f) f_1'.$$

From (4.2), we obtain

$$f' f_1 = R' f_1 + S' - \frac{S}{f_1} f_1'. \tag{4.3}$$

Assume that

$$0 < m(x) \le f(x) \le M(x), \tag{4.4}$$

for all $x \ge n_0$, and suppose inductively that f is increasing on $[n_0, n]$. This means that $f_1' \ge 0$. Then, with our notation convention $m_1 = m(x-1)$ etc., we have:

Theorem 4.1

If $R' \ge 0$, $R' m_1 + S' \ge 0$ and $S \le 0$, then $f' \ge 0$.

Proof

Obvious from (4.3), (4.4) and the above discussion. ∎

Theorem 4.2

Suppose $R' \ge 0$, $S' \ge 0$, $S \ge 0$ and $m_1 m_2 (R' m_1 + S') \ge S(R_1' M_2 + S_1')$. Then $f' \ge 0$.

Proof

Divide (4.3) by f_1 and substitute in by the same rule f_1'. WE obtain

$$f' = \frac{R'f_1 + S'}{f_1} - \frac{S}{f_1^2}f_1' = \frac{R'f_1 + S'}{f_1} - \frac{S}{f_1^2}\left[\frac{R_1'f_2 + S_1'}{f_2} - \frac{S_1}{f_1^2}f_2'\right],$$

and this implies

$$f' = \frac{R'f_1 + S'}{f_1} - \frac{S}{f_1^2}\frac{R_1'f_2 + S_1'}{f_2} + \frac{SS_1}{(f_1f_2)^2}f_2'. \tag{4.5}$$

By the inductive hypothesis, $f_2' = f'(x-2) \geq 0$, and since $SS_1 \geq 0$, it follows that the last term in (4.5) is non-negative. On the other hand,

$$\frac{R'f_1 + S'}{f_1} - \frac{S}{f_1^2}\frac{R_1'f_2 + S_1'}{f_2} \geq 0 \Longleftrightarrow f_1f_2(R'f_1 + S') \geq S(R_1'f_2 + S_1'). \tag{4.6}$$

Since $R', S', S \geq 0$ and $m_1m_2(R'm_1 + S') \geq S(R_1'M_2 + S_1')$, by our assumptions, it follows that (4.6) holds, and we are done. ∎

Now let us show how to apply the above method to some combinatorially relevant numbers.

Recall that the n-th **big Schröder number** r_n is the number of lattice paths from $(0,0)$ to (n,n) with steps $(1,0)$, $(0,1)$ and $(1,1)$ that never rise above the line $y = x$. Equivalently, r_n is the number of lattice paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$, $(1,-1)$ and $(2,0)$ that never fall below the x-axis. For $n = 0$ we put $r_0 := 1$. So, r_n is the number of Motzkin paths whose every plateau, every valley, every terrace and every plain is of even length. Hence, via Proposition 3.6, big Schröder numbers count secondary structures whose unpaired bases form contiguous blocks of even length. The n-th **little Schröder number** is $s_n = \frac{1}{2}r_n$, $s_0 = 1$.

Theorem 4.3

The Schröder numbers are log-convex. The sequences r_n/r_{n-1} and s_n/s_{n-1} increasingly tend to $3+2\sqrt{2}$ as $n \to \infty$.

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Proof

Similarly to the proof of Proposition 3.2, it can be shown (and it is well known) that (r_n) satisfies the following (convolutional) recurrence:

$$r_{n+1} = r_n + \sum_{j=0}^{n} r_j r_{n-j}.$$

This implies that the generating function for (r_n) is given by

$$r(x) = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x}.$$

This, in turn, implies (as in Proposition 3.2) that r_n satisfy the following short recursion

$$r_n = \frac{1}{n+1}[3(2n-1)r_{n-1} - (n-2)r_{n-2}], \tag{4.7}$$

together with the initial conditions $r_0 = 1$, $r_1 = 2$. (There are also combinatorial proofs of (4.7), see [15] or [24].) Divide (4.7) by r_{n-1} and denote r_n/r_{n-1} by $q(n)$. We get

$$q(n) = \frac{3(2n-1)}{n+1} - \frac{n-2}{n+1}\frac{1}{q(n-1)}, \quad n \geq 2, \tag{4.8}$$

with $q(1) = 2$. Then $q(2) = 3$ and $q(3) = 11/3$. Define the function $f : [2, \infty) \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{3}(2x+5) & , \quad x \in [2,3] \\ \frac{1}{x+1}[6x - 3 - \frac{x-2}{f(x-1)}] & , \quad x \geq 3. \end{cases}$$

It is easy to show by induction on n that f is bounded on $[2, n]$, for $n \geq 2$. More precisely, $3 \leq f(x) \leq 6$, for all $x \geq 2$. Also, f is continuous everywhere and differentiable on open intervals $(n, n+1)$, $n \geq 2$. In the notations of Theorem 4.1, we have

$$R(x) = \frac{3(2x-1)}{x+1}, \quad S(x) = -\frac{x-2}{x+1}, \quad m(x) = 3, \quad M(x) = 6.$$

Then

$$R'(x) = \frac{9}{(x+1)^2} \geq 0, \quad S'(x) = -\frac{3}{(x+1)^2}.$$

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Clearly, $S(x) \leq 0$ for $x \geq 2$, and let us check that $R'm_1 + S' \geq 0$. But this is obvious, since

$$\frac{9}{(x+1)^2} \cdot 3 - \frac{3}{(x+1)^2} = \frac{24}{(x+1)^2} \geq 0.$$

Since $f'(x) = 2/3 \geq 0$ for $x \in (2,3)$, it follows from Theorem 4.1 that $f'(x) \geq 0$ for all $x \in (n, n+1)$, $n \geq 2$. Hence, by continuity, f is increasing, so $(q(n))$ is increasing and therefore (r_n) is log-convex. By passing to limit in (4.9) we get the last claim. ∎

Now that we are more familiar with this method, let us show once again (but this time almost automatically) the log-convexity of Motzkin numbers.

Recalling the short recursion (3.9), we define the function $f : [2, \infty) \to \mathbb{R}$ by

$$f(x) = \begin{cases} 2 & , \quad x \in [2,3] \\ \frac{1}{x+2}[2x + 1 + \frac{3(x-1)}{f(x-1)}] & , \quad x \geq 3. \end{cases}$$

Here we have

$$R(x) = \frac{2x+1}{x+2}, \quad S(x) = \frac{3(x-1)}{x+2},$$

so

$$R'(x) = \frac{3}{(x+2)^2} \geq 0, \quad S'(x) = \frac{9}{(x+2)^2} \geq 0.$$

Further, it is easy to check that $2 \leq f(x) \leq 7/2$ for all $x \geq 2$. So, we may take $m(x) = 2$, $M(x) = 7/2$.

Let us check the inequality $m_1 m_2 (R'm_1 + S') \geq S(R_1' M_2 + S_1')$ from Theorem 4.2. But this is equivalent to

$$4\left[2 \cdot \frac{3}{(x+2)^2} + \frac{9}{(x+2)^2}\right] \geq \frac{3x-3}{x+2}\left[\frac{7}{2}\frac{3}{(x+2)^2} + \frac{9}{(x+2)^2}\right],$$

and this is equivalent to

$$1.5x^2 + 61.5x + 117 \geq 0.$$

This last inequality is certainly true for all $x \geq 1$. Hence, by Theorem 4.2, $f' \geq 0$, and this implies the log-convexity of Motzkin numbers. ∎

A **directed animal** of size n is a subset $S \subseteq (\mathbb{N} \cup \{0\})^2$ of cardinality n with the following property: if $p \in S$, then there is a lattice path from $(0,0)$ to p with steps $(1,0)$, $(0,1)$, all of whose vertices lie in S. Let a_n be the number of directed animals of size n. It is well known that the generating function for $(a_n)_{n \geq 1}$ is given by

$$A(x) = \sum_{n \geq 1} a_n x^n = \frac{1}{2} \left(\sqrt{\frac{1+x}{1-3x}} - 1 \right) = x + 2x^2 + 5x^3 + 13x^4 + 35x^5 + \ldots$$

These same numbers appear also as the row sums of Motzkin triangle (see [10] and [4]).

Theorem 4.4

The sequence $(a_n)_{n \geq 1}$ of numbers of directed animals is log-convex. Hence the sequence a_n/a_{n-1} increasingly tends to 3.

Proof

Taking the derivative $A'(x)$ of the generating function $A(x)$ we get

$$(1+x)(1-3x)A'(x) = 2A(x) + 1,$$

and equating coefficients of x^{n-1} above yields the recurrence

$$na_n = 2na_{n-1} + 3(n-2)a_{n-2}, \quad n \geq 3,$$

with $a_1 = 1$, $a_2 = 2$.

The corresponding function for the successive quotients is given by

$$f(x) = 2 + \frac{3(x-2)}{x} \frac{1}{f(x-1)}, \quad x \geq 3,$$

and $f(x) = \frac{x+2}{2}$ on $[2,3]$. It is easy to check that $2 \leq f(x) \leq \frac{7}{2}$.

Here $R(x) = 2$, $S(x) = \frac{3(x-2)}{x} \geq 0$ and $R'(x) = 0$, $S'(x) = \frac{6}{x^2} \geq 0$, and $m = 2$, $M = 7/2$, so the claim follows from Theorem 4.2. ∎

We aplly now our approach to Franel's sequences. Recall that the n-th **Franel number** of order r is defined by

$$S_n^{(r)} = \sum_{k=0}^{n} \binom{n}{k}^r.$$

The numbers $S_n^{(r)}$ satisfy a linear homogenous recurrence of order $\lfloor \frac{r+1}{2} \rfloor$ with polynomial coefficients. For $r = 3$, the recurrence reads as follows ([27]):

$$n^2 S_n^{(3)} = (7n^2 - 7n + 2)S_{n-1}^{(3)} + 8(n-1)^2 S_{n-2}^{(3)}, \tag{4.9}$$

with the initial conditions $S_0^{(3)} = 1$, $S_1^{(3)} = 2$.

Theorem 4.5

The sequence $\left(S_n^{(3)}\right)_{n \geq 0}$ is log-convex. The same is true for the sequence $\left(S_n^{(4)}\right)_{n \geq 0}$.

Proof

Again, taking the quotients $q(n) = S_n^{(3)}/S_{n-1}^{(3)}$ in (4.9), we get

$$q(n) = \frac{7n^2 - 7n + 2}{n^2} + \frac{8(n-1)^2}{n^2} \frac{1}{q(n-1)}, \quad n \geq 2, \tag{4.10}$$

with $q(1) = 2$. Then $q(2) = 5$. Define the function $f : [1, \infty) \to \mathbb{R}$ by

$$f(x) = \begin{cases} 3x - 1 & , \quad x \in [1, 2] \\ \frac{7x^2 - 7x + 2}{x^2} + \frac{8(x-1)^2}{x^2} \frac{1}{f(x-1)}] & , \quad x \geq 2. \end{cases}$$

The function f is bounded. More precisely, $5 \leq f(x) \leq 9$ for all $x \geq 2$. Indeed, f is clearly bounded on $[1, 2]$. On $[2, 3]$ f is given by

$$f(x) = \frac{21x^2 - 41x + 18}{3x^2 - 4x},$$

and one can check easily that $5 \leq f(x) \leq 9$ on this interval. Now, by induction on $n \geq 3$, it is not hard to verify that if $5 \leq f(x) \leq 9$ for $x \in [2, n]$, then also $5 \leq f(x) \leq 9$ for $x \in [n, n+1]$. Also, f is a continuous function. So, here we have, in the notations of Theorem 4.1:

$$R(x) = \frac{7x^2 - 7x + 2}{x^2}, \quad S(x) = \frac{8(x-1)^2}{x^2}$$

and we may take $m(x) = 5$, $M(x) = 9$ for $x \geq 2$. Since $R'(x) = \frac{7x-2}{x^3} \geq 0$, $S'(x) = \frac{16(x-1)}{x^3} \geq 0$ and $S(x) \geq 0$ for $x \geq 2$, it remains only to check the last condition in Theorem 4.2, i.e.

$$25 \left[5 \cdot \frac{7x-2}{x^3} + \frac{16(x-1)}{x^3} \right] \geq \frac{8(x-1)^2}{x^2} \left[\frac{7x-9}{(x-1)^3} \cdot 9 + \frac{16(x-2)}{(x-1)^3} \right].$$

This is equivalent to

$$643x^2 - 1021x + 650 \geq 0,$$

and this last inequality is certainly true for $x \geq 1$.

So, as before, we conclude that f increases, hence $(q(n))_{n \geq 0}$ is an increasing sequence and therefore $\left(S_n^{(3)} \right)_{n \geq 0}$ is log-convex.

Note that $(q(n))$ converges, being increasing and bounded, and so, passing to limit in (4.10) we obtain that

$$\lim_{n \to \infty} \frac{S_n^{(3)}}{S_{n-1}^{(3)}} = 8.$$

A similar proof applies to the case of $\left(S_n^{(4)} \right)_{n \geq 0}$, starting with the recurrence

$$n^3 S_n^{(4)} = 2[6n^3 - 9n^2 + 5n - 10] S_{n-1}^{(4)} + (4n-3)(4n-4)(4n-5) S_{n-2}^{(4)},$$

$S_0^{(4)} = 1$, $S_1^{(4)} = 2$. It turns out that

$$\lim_{n \to \infty} \frac{S_n^{(4)}}{S_{n-1}^{(4)}} = 16.$$

We leave the details to the reader. ∎

As a bit more simple case, let us show the log-behavior of sequences defined by two-term recurrences with constant coefficients.

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Proposition 4.6

Let $(a_n)_{n \geq 0}$ be a positive sequence defined by

$$a_n = C_1 a_{n-1} - C_2 a_{n-2},$$

where C_1, $C_2 > 0$ are constants, and with some initial conditions $a_0 > 0$, $a_1 > 0$. Then the log-behavior of (a_n) is completely determined by the log-behavior of its first three terms. In other words, the sequence (a_n) is log-convex if $a_1^2 \leq a_0 a_2$, and log-concave if $a_1^2 \geq a_0 a_2$.

Proof

Let $q(n) = a_n/a_{n-1}$. The recursion for $q(n)$ is given by

$$q(n) = C_1 - \frac{C_2}{q(n-1)},$$

and the first two values are $q(1) = a_1/a_0$, $q(2) = a_2/a_1$. Set $\Delta = a_0 a_2 - a_1^2$ and define the function $f : [1, \infty) \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{\Delta}{a_0 a_1}(x-1) + \frac{a_1}{a_0} &, \quad x \in [1, 2] \\ C_1 - \frac{C_2}{f(x-1)} &, \quad x \geq 2. \end{cases}$$

f is a linear fractional mapping on any interval $(n, n+1)$, $n \in \mathbb{N}$, and f is continuous.

Suppose first that $\Delta \geq 0$. Then f is obviously increasing on $[1, 2]$ and bounded from below by $f(1) = q(1) = a_1/a_0$. Namely, suppose that $f(x) \geq f(1)$ on $[1, n]$. For $x \in [n, n+1]$ we have

$$f(x) = C_1 - \frac{C_2}{f(x-1)} \geq C_1 - \frac{C_2}{f(1)} = f(2) \geq f(1),$$

and so $f(x)$ is bounded from below on the whole interval $[1, n+1]$. Hence f is bounded from below by $f(1)$. This also shows that f is differentiable on any open interval $(n, n+1)$, $n \in \mathbb{N}$. Here we have $R(x) = C_1$, $S(x) = -C_2$ (so $R'(x) = S'(x) = 0$ for $x \notin \mathbb{N}$) and we may take $m(x) = a_1/a_0$. It follows by Theorem 4.1 that $f'(x) \geq 0$ for all $x \notin \mathbb{N}$. By continuity, f is an increasing function, and hence $f(n) = q(n)$ is an increasing sequence.

If $\Delta \leq 0$, then f decreases on $[1,2]$ and similarly as above f is, in this case, bounded from above by $f(1)$. Since

$$f'(x) = C_2 \frac{f'(x-1)}{f(x-1)},$$

we see that the sign of the derivative is propagated to the right and is determined by its sign on $(1,2)$. Hence, in this case, $f'(x) < 0$. ■

Corollary 4.7

The sequence $(F_{2n+1})_{n\geq 0}$ of odd-indexed Fibonacci numbers is log-convex, while the sequence of even-indexed Fibonacci numbers $(F_{2n})_{n\geq 1}$ is log-concave.

Proof

Let $a_n = F_{2n+1}$. The numbers a_n satisfy the recursion $a_n = 3a_{n-1} - a_{n-2}$, and $a_1^2 = 4 < 5 = a_0 a_2$. The numbers $b_n = F_{2n}$ satisfy the same recursion $b_n = 3b_{n-1} - b_{n-2}$, but this time with initial conditions $b_1 = 1$, $b_2 = 3$. However, $b_2^2 = 9 > 8 = b_1 b_3$. So, the claim follows from Proposition 4.6. (Of course, this Corollary is an immediate consequence of the Cassini identity $F_n^2 - F_{n-1}F_{n+1} = (-1)^n$.) ■

As our next application of the method, we consider some classical orthogonal polynomials. Let $\nu > -1/2$ be a parameter. The **Gegenbauer** (or **ultraspherical**) **polynomials** $C_n^{(\nu)}(t)$ are defined by ([31])

$$nC_n^{(\nu)}(t) = 2t(\nu + n - 1)C_{n-1}^{(\nu)}(t) - (2\nu + n - 2)C_{n-2}^{(\nu)}(t), \quad n \geq 2, \tag{4.11}$$

together with the initial conditions $C_0^{(\nu)}(t) = 1$, $C_1^{(\nu)}(t) = 2\nu t$.

For $\nu = 1/2$, $C_n^{(1/2)}(t) = P_n(t)$ is the **Legendre polynomial** and for $\nu = 1$, $C_n^{(1)}(t) = U_n(t)$ is the **Chebyshev polynomial of the second kind**.

Theorem 4.8

The sequence $\left(C_n^{(\nu)}(t) \right)_{n \geq 0}$ is log-concave for $t \geq 1$, $\nu \geq 1$, and log-convex for $0 < \nu < 1$, $t \geq \max \left\{ \frac{1}{\sqrt{2\nu}}, \frac{1}{\sqrt{2(1-\nu)}} \right\}$.

Proof

Let $q_t(n) = \frac{C_n^{(\nu)}(t)}{C_{n-1}^{(\nu)}(t)}$. Dividing the recursion (4.11) by $nC_{n-1}^{(\nu)}(t)$ we obtain the recursion

$$q_t(n) = 2t\frac{\nu + n - 1}{n} - \frac{2\nu + n - 2}{n}\frac{1}{q_t(n-1)}, n \geq 1 \tag{4.12}$$

with initial condition $q_t(1) = 2\nu t$. Then $q_t(2) = t(\nu + 1) - \frac{1}{2t}$. Now define the function $f_t : [1, \infty) \to \mathbb{R}$ by formula

$$f_t(x) = \begin{cases} \frac{2t^2(1-\nu)-1}{2t}x + \frac{2t^2(3\nu-1)+1}{2t} & , \text{ if } x \in [1,2], \\[2mm] \frac{2t(\nu+x-1)}{x} - \frac{2\nu+x-2}{x}\frac{1}{f_t(x-1)} & , \text{ if } x \geq 2. \end{cases} \tag{4.13}$$

The function $f_t(x)$ is piecewise rational, i.e. it is rational on all segments $[n, n+1]$, $n \in \mathbb{N}$. It is easy to show, by induction on n, that $f_t(x)$ is continuous at the points $x = n$, for all $n \in \mathbb{N}$, and that $f_t(n) = q_t(n)$.

Let us first consider the case $\nu \geq 1$ and $t \geq 1$. We claim that $f_t(x)$ is smooth on all intervals $(n, n+1)$, $n \in \mathbb{N}$. To prove this, it is enough to show that $f_t(x)$ is bounded from below by some positive quantity. Let us check that $f_t(x) \geq \frac{1}{t}$ for all $x \geq 1$.

Since $\nu \geq 1$, $t \geq 1$, the function $f_t(x)$ is non-increasing on the interval $[1, 2]$, so it is enough to show that $f_t(2) \geq \frac{1}{t}$. But this is equivalent to $t(\nu + 1) > \frac{3}{2t}$, or $2t^2(\nu + 1) \geq 3$. Since $\nu \geq 1$, we have $4t^2 \geq 3$, and this is true for all $t \geq 1$.

Suppose now inductively that $f_t(x) \geq \frac{1}{t}$ for all $x \in [1, n]$, and take an $x \in [n, n+1]$, $n \geq 2$. From (4.13) we have

$$f_t(x) \geq \frac{2t(\nu + x - 1)}{x} - \frac{2\nu + x - 2}{x}\frac{1}{\frac{1}{t}} = t,$$

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and $t \geq \frac{1}{t}$ for all $t \geq 1$. So, the function $f_t(x)$ is nowhere zero on $[1, \infty)$, hence it has no poles on $[1, \infty)$ and hence is smooth on all intervals $(n, n+1)$, $n \in \mathbb{N}$.

The function $f_t(x)$ is obviously decreasing on $[1, 2]$. Suppose now that it is decreasing on $[1, n]$. Let $x \in (n, n+1)$. Then

$$f_t'(x) = \frac{2(1-\nu)}{x^2 f_t(x-1)} \left[t f_t(x-1) - 1 \right] + \left(1 + 2\frac{\nu - 1}{x} \right) \frac{f_t'(x-1)}{f_t^2(x-1)}. \tag{4.14}$$

The term in square brackets is positive, since $f_t(x) \geq \frac{1}{t}$, so the whole first term in (4.14) is negative. The second term is negative by the induction hypothesis, hence $f_t'(x) \leq 0$. This completes the step of induction, and we can conclude that the function $f_t(x)$ is decreasing on $[1, \infty)$. Hence, the sequence $\left(C_n^{(\nu)}(t) \right)_{n \geq 0}$ is log-concave for all $\nu \geq 1$, $t \geq 1$.

Let us now consider the case $0 < \nu < 1$, $t \geq \max\left\{ \frac{1}{\sqrt{2\nu}}, \frac{1}{\sqrt{2(1-\nu)}} \right\}$. Obviously, for such values of t, the function $f_t(x)$ is increasing on $[1, 2]$, and $f_t(1) \geq \frac{1}{t}$, hence $f_t(x) \geq \frac{1}{t}$ on $[1, 2]$. It follows easily by induction on n that $f_t(x) \geq \frac{1}{t}$ on $[1, \infty)$, hence $f_t(x)$ is bounded from below by a positive quantity, hence $f_t(x)$ is smooth on all open intervals $(n, n+1)$, $n \in \mathbb{N}$.

Suppose now that $f_t(x)$ is increasing on $[1, n]$ and let $x \in (n, n+1)$, $n \geq 2$. Take a look at (4.14) again. The first term is positive since $f_t(x) \geq \frac{1}{t}$, and the second term is positive by the induction hypothesis for all $x \geq 2$. Hence $f_t'(x) \geq 0$ and $f_t(x)$ is increasing on $[1, n+1]$. The increasing behavior of $f_t(x)$ implies the log-convexity of the sequence $\left(C_n^{(\nu)}(t) \right)_{n \geq 0}$. ■

In Fig. 5 we can see the areas of log-convexity (denoted by Λ) and log-concavity (Ξ) of sequences $\left(C_n^{(\nu)}(t) \right)_{n \geq 0}$. Our analysis is not sufficient to determine the logarithmic behavior of sequences $\left(C_n^{(\nu)}(t) \right)_{n \geq 0}$ whose parameters (ν, t) fall into the areas denoted by I and II.
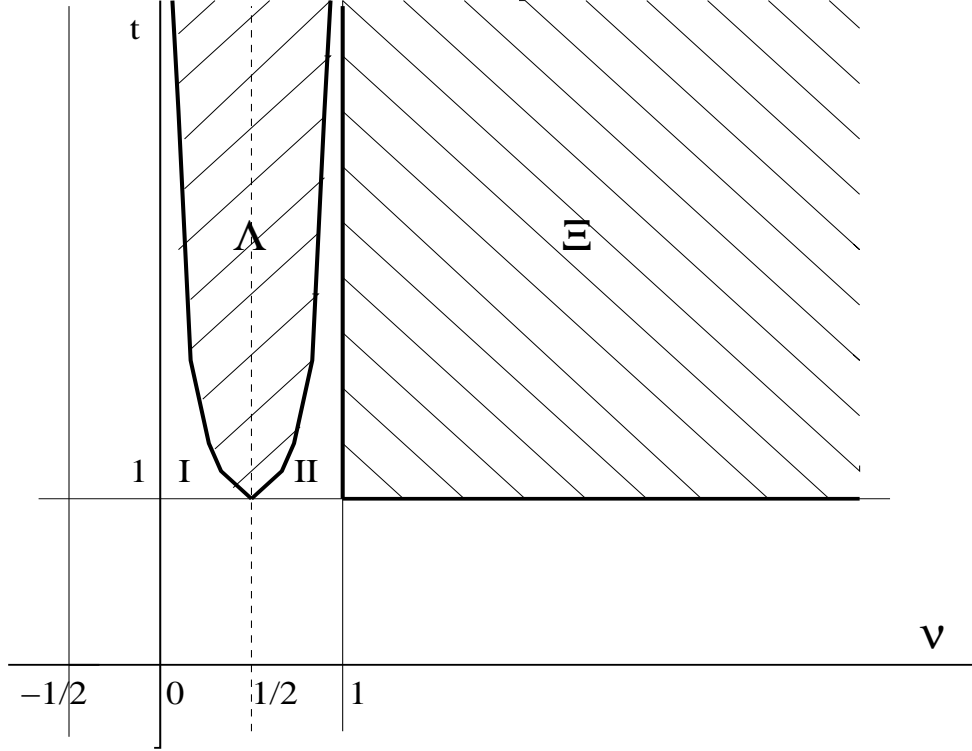
Figure 5: Areas of log-convexity (Λ) and log-concavity (Ξ) for sequences of values of Gegenbauer polynomials

Remark 4.9

So far we have restricted our definitions of log-convexity and log-concavity to positive (or perhaps non-negative) sequences, since most combinatorial sequences have only non-negative terms. The definitions, however, make sense even if we include negative numbers.

Corollary 4.10

The sequence $\left(C_n^{(\nu)}(t) \right)_{n \geq 0}$ is log-concave for $|t| \geq 1$, $\nu \geq 1$, and log-convex for $0 < \nu < 1$, $|t| \geq \max\left\{ \frac{1}{\sqrt{2\nu}}, \frac{1}{\sqrt{2(1-\nu)}} \right\}$.

Proof

It follows from Theorem 4.7 and the relation

$$C_n^{(\nu)}(-t) = (-1)^n C_n^{(\nu)}(t),$$

valid for all $\nu \neq 0$ and all t. ■

Corollary 4.11

The sequence $(U_n(t))_{n\geq 0}$ of the values of Chebyshev polynomials of the second kind is log-concave for $|t| \geq 1$. In particular, $F_{2n} = U_n(3/2)$ is log-concave.

Proof

Follows from Corollary 4.10, since $U_n(t) = C_n^{(1)}(t)$. ■

Corollary 4.12

The sequence $(P_n(t))_{n\geq 0}$ of the values of Legendre polynomials is log-convex for $|t| \geq 1$ and log-concave for $|t| \leq 1$.

Proof

$P_n(t) = C_n^{(1/2)}(t)$ and log-convexity follows from Corollary 4.10. The log-concavity follows from the well known fact that $|P_n(t)| \leq 1$ for $|t| \leq 1$ and from the inequality ([2], [30])

$$P_n^2(t) - P_{n-1}(t)P_{n+1}(t) \geq \frac{1 - P_n^2(t)}{(2n-1)(n+1)},$$

valid for $|t| \leq 1$. ■

Our results on Legendre polynomials complement and generalize in some respects those of Turán and Szegö ([32]), who considered determinants of the form $\begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_{n-1}(x) & P_n(x) \end{vmatrix}$.

Now, a combinatorial consequence is in order. The n-th **central Delannoy number** $D(n)$ counts

the lattice paths in (x, y) coordinate plane from $(0, 0)$ to (n, n) with steps $(1, 0)$, $(0, 1)$ and $(1, 1)$. (Such paths are also known as **"king's paths"**.) The generating function of $(D(n))_{n \geq 0}$ is given by $D(x) = \frac{1}{\sqrt{1 - 6x + x^2}}$ ([27]). This implies easily that $D(n)$'s satisfy the recurrence (4.11) for $\nu = 1/2$ and $t = 3$ and $D(0) = 1$, $D(1) = 3$. Hence $D(n) = P_n(3)$, the value of the n-th Legendre polynomial P_n at $t = 3$.

Corollary 4.13.

The sequence $D(n)$ of Delannoy numbers is log-convex and $\lim_{n \to \infty} \frac{D(n)}{D(n-1)} = 3 + 2\sqrt{2}$.

Proof

The log-convexity follows from Corollary 4.12. Hence $q_3(n) = D(n)/D(n-1)$ is increasing. From (4.12) for $\nu = 1/2$ and $t = 3$ it follows easily by induction that $q_3(n) \leq 6$. Passing to limit, we obtain the second claim. ∎ ■

Remark 4.14

No wonder that the above limit coincides with the limit for Schröder numbers from Theorem 4.3. This can be explained by showing that

$$r_n = \frac{P_n'(3)}{P_n'(1)}.$$

(For a proof, see e.g. [11].)

The derivatives of ultraspherical polynomials behave log-concavely. More precisely, the following is valid.

Theorem 4.15

The sequence $\left(C_n^{(\nu)'}(t) \right)_{n \geq 0}$ is log-concave, for all $\nu > 0$, $t \geq 1$.

Proof

Deriving the recurrence (4.11) with respect to t and using the relation

$$2(\nu + n - 1)C_{n-1}^{(\nu)}(t) = C_n^{(\nu)'}(t) - C_{n-2}^{(\nu)'}(t),$$

we get the two-term recursion for derivatives:

$$C_n^{(\nu)'}(t) = 2t\frac{\nu + n - 1}{n - 1}C_{n-1}^{(\nu)'}(t) - \frac{2\nu + n - 1}{n - 1}C_{n-2}^{(\nu)'}(t),$$

which starts with $C_0^{(\nu)'}(t) = 0$, $C_1^{(\nu)'}(t) = 2\nu$, $C_2^{(\nu)'}(t) = 4\nu(\nu + 1)t$ and $C_3^{(\nu)'}(t) = 4\nu(\nu+1)(\nu+2)t^2 - 2\nu(\nu + 1)$. Hence, the successive quotients $q_t(n) = C_n^{(\nu)'}(t)/C_{n-1}^{(\nu)'}(t)$ satisfy the recurrence

$$q_t(n) = 2t\frac{\nu + n - 1}{n - 1} - \frac{2\nu + n - 1}{n - 1}\frac{1}{q_t(n - 1)}, n \geq 3,$$

starting with $q_t(2) = 2(\nu + 1)t$, $q_t(3) = (\nu + 2)t - \frac{1}{2t}$. The appropriate function $f_t : [2, \infty) \to \mathbb{R}$ is defined by

$$f_t(x) = \begin{cases} -\frac{1}{2t}(2\nu t^2 + 1)x + (4\nu + 1)t + \frac{1}{t} & , \text{ if } x \in [2, 3], \\ \frac{2t(\nu + x - 1)}{x - 1} - \frac{2\nu + x - 1}{x - 1}\frac{1}{f_t(x - 1)} & , \text{ if } x \geq 3. \end{cases}$$

The function $f_t(x)$ is obviously decreasing on $[2, 3]$ for all $\nu > 0$, $t \geq 1$. Also, $f_t(x)$ is continuous in all points $x = n$, for $n \in \mathbb{N}$, and $f_t(n) = q_t(n)$. As before, it is not hard to check that $f_t(x) \geq \frac{1}{t}$, for all $x \geq 2$. Hence, the derivative $f_t'(x)$ exists for any $x \in (n, n + 1)$, $n \geq 2$.

$$f_t'(x) = \frac{2\nu}{(x - 1)^2 f_t(x - 1)}[1 - tf_t(x - 1)] + \left(1 + \frac{2\nu}{x - 1}\right)\frac{f_t'(x - 1)}{f_t^2(x - 1)}.$$

This derivative is obviously non-positive on $(2, 3)$. Suppose inductively that $f_t'(x - 1) \leq 0$. The first term above is negative since $f_t(x) \geq \frac{1}{t}$, and the second term is negative by the induction hypothesis. Hence, $f_t(x)$ is decreasing on $(n, n + 1)$, and then, by continuity, on $[1, n + 1]$. This completes the inductive step and hence $q_t(n) = f_t(n)$ is also decreasing. ■

We conclude our review of orthogonal polynomials with Laguerre polynomials. The (ordinary) **Laguerre polynomials** satisfy the recursion

$$nL_n(t) = (2n - 1 - t)L_{n-1}(t) - (n - 1)L_{n-2}(t)$$

with the initial conditions $L_0(t) = 1$, $L_1(t) = 1 - t$.

Theorem 4.16

The sequence $(L_n(t))_{n \geq 0}$ is log-concave for all $t \leq 0$. The ratio $L_n(t)/L_{n-1}(t)$ is greater than one for

all $t < 0$ and it tends to 1 decreasingly as $n \to \infty$.

Proof

Dividing the above recursion by $nL_{n-1}(t)$ and denoting $\frac{L_n(t)}{L_{n-1}(t)}$ by $q_t(n)$, we get a recursion for $q_t(n)$

$$q_t(n) = \frac{2n - 1 - t}{n} - \frac{n-1}{n}\frac{1}{q_t(n-1)} \tag{4.15}$$

with $q_t(1) = 1 - t$. By computing $q_t(2) = \frac{t^2 - 4t + 2}{2(1-t)}$, we see that $q_t(1) \geq q_t(2)$, for all $t \leq 0$. Define

$f_t : [1, \infty) \to \mathbb{R}$ by formula

$$f_t(x) = \begin{cases} -\frac{t^2}{2(t-1)}x + \frac{3t^2 - 4t + 2}{2(t-1)} & \text{, if } x \in [1, 2], \\ \frac{2x-1-t}{x} - \frac{x-1}{x}\frac{1}{f_t(x-1)} & \text{, if } x \geq 2. \end{cases}$$

This function is, again, piecewise rational, continuous for $x = n \in \mathbb{N}$ and $f_t(n) = q_t(n)$ for all $n \in \mathbb{N}$.

By induction on $n \geq 2$, one can easily check that, for $x \in [1, n]$, we have $1 \leq f_t(x) \leq 1 - t$ for all $t \leq 0$.

Hence,

$$1 \leq f_t(x) \leq 1 - t, \quad x \geq 1, t \leq 0. \tag{4.16}$$

By computing the derivative, we see that $f_t'(x) \leq 0$ for $x \in (1, 2)$, and for $x \in (n, n+1)$, $n \geq 2$ we find

$$f_t'(x) = \frac{1+t}{x^2} - \frac{1}{x^2}\frac{1}{f_t(x-1)} + \left(1 - \frac{1}{x}\right)\frac{f_t'(x-1)}{f_t^2(x-1)}.$$

Suppose $f_t'(x-1) \leq 0$. Then for $t \leq -1$, all three terms on the right hand side are negative, hence

$f_t'(x) \leq 0$. For $-1 < t \leq 0$, the second and third terms are still negative, but the first term is positive.

The claim will follow if we prove

$$\frac{1+t}{x^2} - \frac{1}{x^2}\frac{1}{f_t(x-1)} \leq 0.$$

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But this is equivalent to $(1+t)f_t(x-1)-1 \leq 0$, and this reduces to $f_t(x-1) \leq \frac{1}{1+t}$. But $f_t(x) \leq 1-t$, and for $-1 < t \leq 0$ we have $1-t \leq \frac{1}{1+t}$. Hence, the sum of the first two terms is negative, and $f'_t(x) \leq 0$. By continuity, f_t is decreasing, and so $q_t(n)$ is decreasing. Since by (4.16) this sequence is also bounded, it is convergent and passing to limit in (4.15) we get that this limit is 1 as claimed. ∎

Some of our results on orthogonal polynomials, in particular those concerning Legendre and Laguerre polynomials, have been already known (see, e.g. [14]), but here we derived them in a simple and unified manner, almost automatically.

So far we have been considering only the two-term recurrences. The "calculus" method, however, works as well for higher order recurrences, as the following example shows.

Let $L(n)$ be the number of graphs on the vertex set $[n]$, whose every component is a cycle, and define $L(0) := 1$. Then $L(1) = 1$, $L(2) = 2$, $L(3) = 5$, $L(4) = 17$ etc. The following recurrence holds ([27], Ex. 5.22):

$$L(n) = nL(n-1) - \binom{n-1}{2} L(n-3), \quad n \geq 3. \tag{4.17}$$

Theorem 4.17

The sequence $(L(n))_{n \geq 0}$ is log-convex.

Proof

Let $q(n) = L(n)/L(n-1)$. From (4.17), dividing with $L(n-1)$, we obtain

$$q(n) = n - \binom{n-1}{2} \frac{1}{q(n-1)q(n-2)}, \quad n \geq 3, \tag{4.18}$$

with the initial conditions $q(1) = 1$, $q(2) = 2$. The value $q(3)$ is equal to $5/2$. We claim that $(q(n))_{n \geq 1}$

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is an increasing sequence. To this end, define the function $f : [1, \infty) \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{, if } x \in [1,2], \\ \frac{x+2}{2} & \text{, if } x \in [2,3], \\ x - \frac{(x-1)(x-2)}{2}\frac{1}{f(x-1)f(x-2)} & \text{, if } x \geq 3. \end{cases} \qquad (4.19)$$

The function f is obviously continuous, rational on any $[n, n+1]$, $n \in \mathbb{N}$, without poles on $[n, n+1]$ and clearly increasing on $[1, 3]$. Let us first prove an a priori bound

$$f(x) \geq \sqrt{x+1}, \quad x \geq 2. \qquad (4.20)$$

This is clearly true for $x \in [2, 3]$. Suppose that it is true for $x \in [2, n]$, and let $x \in [n, n+1]$. Then from (4.20) we have

$$f(x) \geq x - \frac{(x-1)(x-2)}{2}\frac{1}{\sqrt{x}\sqrt{x-1}} = x - \frac{(x-2)\sqrt{x-1}}{2\sqrt{x}},$$

and it is easy to check that

$$x - \frac{(x-2)\sqrt{x-1}}{2\sqrt{x}} \geq \sqrt{x+1}, \quad x \geq 2.$$

The function f is differentiable for all $x \geq 1$, $x \notin \mathbb{N}$. Let $x \in (n, n+1)$ for some $n \geq 2$ and denote $f_1 = f(x-1)$, $f_2 = f(x-2)$. Suppose inductively that $f_1' \geq 0$, $f_2' \geq 0$. Then

$$f'(x) = 1 - \frac{2x-3}{2}\frac{1}{f_1 f_2} + \frac{(x-1)(x-2)}{2}\frac{f_1' f_2 + f_1 f_2'}{(f_1 f_2)^2}.$$

The last term is positive, and from (4.21) it follows that

$$1 - \frac{2x-3}{2}\frac{1}{f_1 f_2} \geq 0 \iff f_1 f_2 \geq \sqrt{x}\sqrt{x-1} = \sqrt{x^2 - x} \geq x - \frac{3}{2},$$

for $x \geq 2$. Hence, $f'(x) \geq 0$ and theorem is proved. ∎

Consider now a general three-term recurrence of the form

$$a(n) = R(n)a(n-1) + S(n)a(n-2) + T(n)a(n-3), \quad n \geq 4. \qquad (4.21)$$

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Dividing this recurrence by $a(n-1)$ and denoting $q(n) = a(n)/a(n-1)$, we obtain the recurrence for $q(n)$:

$$q(n) = R(n) + \frac{S(n)}{q(n-1)} + \frac{T(n)}{q(n-1)q(n-2)}, \quad n \geq 3, \tag{4.22}$$

with some given initial conditions, say, $q(1) = b_1$, $q(2) = b_2$, $b_2 > b_1 > 0$.

Suppose we want to prove that $(q(n))_{n \geq 1}$ is an increasing sequence. Again, form the function $f : [1, \infty) \to \mathbb{R}$ mimicking the rule (4.22) and starting with the linear functions on $[1, 2]$ and $[2, 3]$, connecting the points $(1, b_1)$ and $(2, b_2)$ and the points $(2, b_2)$ and $(3, b_3)$, respectively, where $b_3 = q(3) = R(3) + \frac{S(3)}{b_2} + \frac{T(3)}{b_1 b_2}$ (supposing, of course, $b_3 \geq b_2$). For $x \geq 3$, the function f is defined by replacing q by f and n by x in (4.22).

$$f(x) = R(x) + \frac{S(x)}{f(x-1)} + \frac{T(x)}{f(x-1)f(x-2)}, \tag{4.23}$$

for $x \geq 3$ (or $x \geq n_0 + 1$, for some $n_0 \in \mathbb{N}$). For a fixed x we write simply $f(x) = f$, $R(x) = R$, $S(x) = S$, $T(x) = T$, $f(x-1) = f_1$, $f(x-2) = f_2$, etc. So, we write (4.23) simply as

$$f = R + \frac{S}{f_1} + \frac{T}{f_1 f_2} \iff f_1 f_2 f = R f_1 f_2 + S f_1 + T. \tag{4.24}$$

We assume that R, S and T are "good enough" functions, in the sense that f is differentiable on open intervals $(n, n+1)$ for integers $n \geq 2$ (or $n \geq n_0$). This is usually a consequence of an a priori bound of the type

$$0 < m(x) \leq f(x) \leq M(x), \tag{4.25}$$

for some well-behaved functions $m(x)$ and $M(x)$.

Taking the derivative d/dx of both sides of (4.24) gives us

$$f' = \frac{1}{f_1 f_2} \left[F - \frac{S f_2 + T}{f_1} f_1' - \frac{T}{f_2} f_2' \right], \tag{4.26}$$

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where

$$F = R'f_1f_2 + S'f_2 + T'. \tag{4.27}$$

Suppose that $R, S, T \geq 0$ and R', S', $T' \geq 0$. By substituting the analogous expressions for f_1' and

f_2' in (4.26) and supposing inductively that f_2', f_3' and $f_4' \geq 0$ we obtain (denoting $F_1 = F(x-1)$,

$R_1 = R(x-1)$, etc.) that:

$$f_1 f_2 f' = F - \frac{Sf_2 + T}{f_1 f_2 f_3} F_1 - \frac{T}{f_2 f_3 f_4} F_2 + Bf_2' + Cf_3' + Df_4',$$

where B, C and D are non-negative. To prove that $f' \geq 0$, assuming inductively that f_2', f_3' and

$f_4' \geq 0$, it is enough to prove the following:

$$F - \frac{Sf_2 + T}{f_1 f_2 f_3} F_1 - \frac{T}{f_2 f_3 f_4} F_2 \geq 0,$$

or equivalently

$$f_1 f_2 f_3 f_4 F \geq (Sf_2 + T)f_4 F_1 + Tf_1 F_2. \tag{4.28}$$

Taking into account (4.25), it suffices to prove a stronger inequality

$$m^4 F \geq (MS + T)MF_1 + MTF_2 \tag{4.29}$$

for all $x \geq n_0$, for some $n_0 \in \mathbb{N}$, and to check that $f'(x) \geq 0$ for $x \in (k, k+1)$, $k < n_0$.

Let us now consider a concrete example. A **Baxter permutation** is defined in [27]. The numbers

$B(n)$ of Baxter permutations in Σ_n satisfy the recurrence

$$
\begin{aligned}
(n+1)(n+2)(n+3)(3n-2)B(n) = \; & 2(n+1)(9n^3 + 3n^2 - 4n + 4)B(n-1) \\
& + (3n-1)(n-2)(15n^2 - 5n - 14)B(n-2) \\
& + 8(3n+1)(n-2)^2(n-3)B(n-3), n \geq 4, \quad (4.30)
\end{aligned}
$$

together with the initial conditions $B(0) = 1$, $B(1) = 1$, $B(2) = 2$, $B(3) = 6$.

Theorem 4.18

The numbers $B(n)$ of Baxter permutations are log-convex. The limit $\lim_{n\to\infty} B(n)/B(n-1)$ exists and is equal to 8.

Proof

Let $q(n) = B(n)/B(n-1)$. From (4.30) we form the recurrence for $q(n)$'s, and then according to the initial values $q(1) = 1$, $q(2) = 2$, $q(3) = 3$, we form the function $f : [1, \infty) \to \mathbb{R}$ defined by $f(x) = x$ on $[1, 3]$ and for $x \geq 3$ by the rule

$$
\begin{aligned}
(x + 1)(x + 2)(x + 3)(3x - 2)f(x) \;=\; & 2(x + 1)(9x^3 + 3x^2 - 4x + 4) \\
& + \frac{(3x - 1)(x - 2)(15x^2 - 5x - 14)}{f(x - 1)} \\
& + \frac{8(3x - 1)(x - 2)^2(x - 3)}{f(x - 1)f(x - 2)}.
\end{aligned}
\tag{4.31}
$$

This, written in the form of (4.23), yields to conclude that R, S, T are positive rational functions with no poles on $[1, \infty)$. We also see that $R(x) \nearrow 6$, $S(x) \nearrow 15$, $T(x) \nearrow 8$ as $x \to \infty$.

It is apparent from (4.31) that $f(x)$ is a piecewise rational function, i.e. rational on intervals $[n, n+1]$, $n \in \mathbb{N}$. For example, for $x \in [3, 4]$,

$$
f(x) = \frac{18x^5 + 51x^4 - 122x^3 - 87x^2 + 200x + 12}{(x - 1)(x + 1)(x + 2)(x + 3)(3x - 2)}.
$$

Clearly, $f(x)$ is continuous everywhere. It can be checked, by induction on n, that the function f is bounded. More precisely,

$$
7 \leq f(x) \leq 9, \quad x \geq 47.
\tag{4.32}
$$

We want to prove that f is an increasing function. From the above a priori bound, it follows that f is differentiable on all open intervals $(n, n+1)$, $n \in \mathbb{N}$. The non-negativity of $f'(x)$ can be checked for $x \leq 49$, using e.g. *Mathematica*, and for $x \geq 49$, it follows from the stronger inequality,

$$
7^4 F \geq (9 \cdot 15 + 8) \cdot 9F_1 + 9 \cdot 8F_2,
$$

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obtained by substituting appropriate values for m, M in (4.29). As this inequality is true for all $x \geq 9$, the first claim follows. The second claim follows passing to the limit in (4.30). ■

Many other three-or-higher-term recurrences can be investigated by this method. Let us only mention that the four-term recurrence (3.15) for the numbers $S^{(1)}(n)$ of secondary structures of rank 1 can also be shown to be log-convex by this "calculus" method. The details (rather tedious) are given in [11], [13]. Another example is the number S_n of $n \times n$ symmetric matrices with entries 0, 1, 2, whose sums of all rows and all columns are equal to 2. These numbers satisfy ([27]) the following recurrence

$$S_n = (2n - 1)S_{n-1} - (n - 1)(n - 2)S_{n-2} - (n - 1)(n - 2)S_{n-3} + \frac{1}{2}(n - 1)(n - 2)(n - 3)S_{n-4},$$

starting with $S_0 = 1$, $S_1 = 1$, $S_2 = 3$, $S_3 = 11$. The sequence $(S_n)_{n \geq 0}$ is also log-convex.

Finally, as it should be clear by now, this method applies to any P-recursive sequence $(a(n))$, satisfying a recurrence of the form

$$Q(n)a(n) = P_d(n)a(n - 1) + P_{d-1}(n)a(n - 2) + \ldots + P_0(n)a(n - d - 1), \quad n \geq d + 1, \qquad (4.33)$$

where $d \geq 0$ is an integer and $P_0, \ldots, P_d, Q$ real polynomials, $Q > 0$. The corresponding function for the successive quotients $q(n) = \frac{a(n)}{a(n-1)}$ is given by the functional equation

$$Q(x)f(x) = P_d(x) + \frac{P_{d-1}(x)}{f(x - 1)} + \ldots + \frac{P_0(x)}{f(x - 1) \ldots f(x - d)}, \quad x \geq d + 1. \qquad (4.34)$$

As we have seen, the most important thing in this approach is to express the derivative $f'(x)$ in terms of previous derivatives. So, fix a point $x \in (n, n + 1)$, $n > d$, and write as before for short $f = f(x)$, $f_j = f(x - j)$, $j = 1, 2, \ldots$, $P_i = P_i(x)$, $Q_i = Q_i(x)$, $i = 0, 1, \ldots, d$. Then (4.34) can be written as

$$Qf = \sum_{i=0}^{d} \frac{P_i}{f_1 f_2 \ldots f_{d-i}},$$

or by denoting the product of all values by Π, i.e. $\Pi = f_1 f_2 \ldots f_d$, and by $\Pi_j = f_1 f_2 \ldots f_j$ the partial

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products (so $\Pi_d = \Pi$ and $\Pi_0 = 1$), as

$$\Pi Q f = \sum_{i=o}^{d} (f_{d-i+1} \ldots f_d) P_i. \tag{4.35}$$

Taking the derivative d/dx of both sides in (4.35), after some manipulations, we obtain the following formula (in terms of Wronskians):

$$f' = \frac{1}{Q^2} \sum_{i=0}^{d} \begin{vmatrix} Q & P_i \\ Q' & P_i' \end{vmatrix} \frac{1}{\Pi_{d-i}} - \frac{1}{Q} \sum_{i=0}^{d} \frac{P_i \Pi_{d-i}'}{\Pi_{d-i}^2}. \tag{4.36}$$

In particular, for $d = 1$ this reduces to

$$f' = \frac{1}{Q^2 f_1} \begin{vmatrix} Q & P_0 \\ Q' & P_0' \end{vmatrix} + \frac{1}{Q^2} \begin{vmatrix} Q & P_1 \\ Q' & P_1' \end{vmatrix} - \frac{P_0}{Q f_1^2} f_1'. \tag{4.37}$$

With a priori bounds $0 < m(x) \le f(x) \le M(x)$ and by substituting f_1' in (4.37), one gets almost instant proofs of log-behavior. For example, if $P_0 \le 0$ and if we want to prove the log-convexity, hence assuming $f_1' \ge 0$, then if the first Wronskian W_0 in (4.37) is positive, we only have to check $\frac{1}{M} W_0 + W_1 \ge 0$, and check that f increases at the beginning.

Of course, not every (combinatorially relevant) sequence satisfying a recurrence of this type can be expected to have a reasonable log-behavior; it is enough to recall here the sequences $e_k(n)$ from Section 2, whose log-behavior is rather chaotic for $k \ge 3$.

Let us only mention here that the log-convexity of secondary structure numbers of general rank l can also be proved by calculus method, using the explicit formulae from Proposition 3.9 and formula (4.36). The details will appear elsewhere.

As a finall remark, note that our approach applies also to linear nonhomogeneous recurrences for positive numbers. So, for example, let $(a(n))$ be given by the linear recurrence of the first order

$$a(n) = R(n)a(n-1) + S(n). \tag{4.38}$$

Consider the quotients $q(n) = \frac{a(n)}{a(n-1)}$ and note that

$$a(n) = q(n)q(n-1)\ldots q(2)a(1), \quad n \geq 2. \tag{4.39}$$

Then, dividing (4.38) by $a(n-1)$ we obtain a (long) recurrence for $q(n)$'s:

$$q(n) = R(n) + \frac{S(n)}{q(n-1)q(n-2)\ldots q(2)a(1)}. \tag{4.40}$$

To get a short recurrence for $q(n)$'s, substitute for $a(n)$ and $a(n-1)$ the corresponding products (4.39) in (4.38) ($n \geq 3$):

$$q(n)q(n-1)\ldots q(2)a(1) = R(n)q(n-1)\ldots q(2)a(1) + S(n) =$$

$$R(n)\frac{q(n)q(n-1)\ldots q(2)a(1)}{q(n)} + S(n)\frac{q(n)q(n-1)\ldots q(2)a(1)}{q(n)q(n-1)\ldots q(2)a(1)}.$$

From there we get

$$\frac{1}{q(n)q(n-1)\ldots q(2)a(1)} = \frac{1}{S(n)}\left[1 - \frac{R(n)}{q(n)}\right],$$

and then

$$\frac{1}{q(n-1)\ldots q(2)a(1)} = \frac{1}{S(n-1)}\left[1 - \frac{R(n-1)}{q(n-1)}\right]. \tag{4.41}$$

Substituting (4.41) in (4.40) yields a short recursion for $q(n)$'s:

$$q(n) = R(n) + \frac{S(n)}{S(n-1)} - \frac{R(n-1)S(n)}{S(n-1)}\frac{1}{q(n-1)}.$$

Similarly, for a second order linear recurrence

$$a(n) = R(n)a(n-1) + S(n)a(n-2) + T(n),$$

we obtain

$$q(n) = R(n) + \frac{S(n)}{q(n-1)} + \frac{T(n)}{T(n-1)}\left[1 - \frac{R(n-1)}{q(n-1)} - \frac{S(n-1)}{q(n-1)q(n-2)}\right].$$

Then we can proceed as before.

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5 Calculus method in two variables

We shall outline our method for non-negative sequences $a(n, k)$ in two integer variables $n, k \geq 0$ (or $n \geq n_0$, $k \geq k_0$). Suppose (as often in combinatorics) that the numbers $a(n, k)$ satisfy a two-term recurrence of the form

$$a(n, k) = R(n, k)a(n - 1, k - 1) + S(n, k)a(n - 1, k), \tag{5.1}$$

with some known functions R and S, together with some initial values, usually of the type $a(0, 0) = a$, $a(1, 0) = b$, $a(1, 1) = c$. Suppose we want to prove that the sequence $a(n, k)$ is log-concave in k, i.e. that $a(n, k)^2 \geq a(n, k-1)a(n, k+1)$, for all n, k. Here is what we do. Write down (5.1) with k replaced by $k - 1$:

$$a(n, k - 1) = R(n, k - 1)a(n - 1, k - 2) + S(n, k - 1)a(n - 1, k - 1). \tag{5.2}$$

Denote

$$q(n, k) = \frac{a(n, k)}{a(n, k - 1)}, \tag{5.3}$$

and divide (5.1) by (5.2) (always assuming we do not divide by zero).

$$
\begin{aligned}
q(n, k) &= \frac{R(n, k)a(n - 1, k - 1) + S(n, k)a(n - 1, k)}{R(n, k - 1)a(n - 1, k - 2) + S(n, k - 1)a(n - 1, k - 1)} = \frac{R(n, k) + S(n, k)q(n - 1, k)}{\frac{R(n, k-1)}{q(n-1, k-1)} + S(n, k - 1)} \\
&= q(n - 1, k - 1)\frac{R(n, k) + S(n, k)q(n - 1, k)}{R(n, k - 1) + S(n, k - 1)q(n - 1, k - 1)}.
\end{aligned}
$$

Equivalently,

$$q(n, k)[R(n, k - 1) + S(n, k - 1)q(n - 1, k - 1)] = q(n - 1, k - 1)[R(n, k) + S(n, k)q(n - 1, k)]. \tag{5.4}$$

The log-concavity of $a(n, k)$'s is equivalent to $q(n, k) \geq q(n, k + 1)$, for any fixed n and all k. The idea is again to pass to a "continuation" of (5.4) by letting $n \to x$, $k \to y$, $q \to f$ and obtaining the functional equation

$$f(x, y)[R(x, y - 1) + S(x, y - 1)f(x - 1, y - 1)] = f(x - 1, y - 1)[R(x, y) + S(x, y)f(x - 1, y)]. \tag{5.5}$$

We assume that R and S are "good enough" functions, in the sense that f is continuous everywhere and smooth on open cells $(n, n+1) \times (m, m+1)$, for all m, n. What we want to prove is that f is decreasing in y for any fixed x. Fix a point (x, y) in an open cell $Q = (n, n+1) \times (m, m+1)$, and prove inductively that

$$\frac{\partial f}{\partial y}(x, y) \leq 0. \tag{5.6}$$

For the fixed pair (x, y) write for short $f_{ij} = f(x - i, y - j)$, for $i, j = 0, 1, 2, \ldots$, and similarly for R and S. So, $f_{00} = f(x, y)$, $R_{01} = R(x, y - 1)$ etc. In this notation, (5.5) can be written as

$$f[R_{01} + S_{01}f_{11}] = f_{11}[R + Sf_{10}]. \tag{5.7}$$

Now take the partial derivative $\partial/\partial y$ of both sides in (5.7). We have

$$\frac{\partial f}{\partial y}[R_{01} + S_{01}f_{11}] + f\left[\frac{\partial R_{10}}{\partial y} + f_{11}\frac{\partial S_{01}}{\partial y} + S_{01}\frac{\partial f_{11}}{\partial y}\right] = \frac{\partial f_{11}}{\partial y}[R + Sf_{10}] + f_{11}\left[\frac{\partial R}{\partial y} + f_{10}\frac{\partial S}{\partial y} + S\frac{\partial f_{10}}{\partial y}\right].$$

$$\frac{\partial f}{\partial y}[R_{01} + S_{01}f_{11}] = f_{11}\left[\frac{\partial R}{\partial y} + f_{10}\frac{\partial S}{\partial y}\right] - \left[\frac{\partial R_{10}}{\partial y} + f_{11}\frac{\partial S_{01}}{\partial y}\right]f + \frac{\partial f_{10}}{\partial y}f_{11}S + \frac{\partial f_{11}}{\partial y}[R + Sf_{10} - S_{01}f].$$

Substituting here f from (5.7), we get

$$\frac{\partial f}{\partial y}[R_{01} + S_{01}f_{11}] = \frac{f_{11}}{R_{01} + S_{01}f_{11}}\left[(R_{01} + S_{01}f_{11})(\frac{\partial R}{\partial y} + f_{10}\frac{\partial S}{\partial y}) - (R + Sf_{10})(\frac{\partial R_{01}}{\partial y} + f_{11}\frac{\partial S_{01}}{\partial y}\right]$$
$$+ f_{11}S\frac{\partial f_{10}}{\partial y} + \frac{(R + Sf_{1,0})R_{01}}{R_{01} + S_{01}f_{11}}\frac{\partial f_{11}}{\partial y}. \tag{5.8}$$

Assume that R and S are positive. Hence if f starts with some positive values, then f can be considered positive, too. Suppose inductively that

$$\frac{\partial f_{10}}{\partial y}, \frac{\partial f_{11}}{\partial y} \leq 0.$$

Then the last two terms in (5.8) are negative and to prove (5.6), it is enough to prove that the first term is negative, too. In other words, to conclude (inductively) that (5.6) holds, it is enough to prove that the "free" term is non-positive, i.e. that

$$F = (R_{01} + S_{01}f_{11})\left(\frac{\partial R}{\partial y} + f_{10}\frac{\partial S}{\partial y}\right) - (R + Sf_{10})\left(\frac{\partial R_{01}}{\partial y} + f_{11}\frac{\partial S_{01}}{\partial y}\right) \leq 0. \tag{5.9}$$

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So, if we can check that f begins decreasingly in y and assuming inductively that f is decreasing in y, then by (5.9) we can conclude that f is decreasing in y at the point (x, y) and then, by continuity, that f is decreasing in y everywhere.

Note that the following inequalities imply (5.9) (simply by comparing similar terms):

$$R_{01}\frac{\partial R}{\partial y} \leq R\frac{\partial R_{01}}{\partial y}, S_{01}\frac{\partial R}{\partial y} \leq R\frac{\partial S_{01}}{\partial y}, R_{01}\frac{\partial S}{\partial y} \leq S\frac{\partial R_{01}}{\partial y}, S_{01}\frac{\partial S}{\partial y} \leq S\frac{\partial S_{01}}{\partial y}.$$

In terms of Wronskians, writing G' for $\frac{\partial G}{\partial y}$, these inequalities can be written in the form:

$$\begin{vmatrix} R_{01} & R \\ R'_{01} & R' \end{vmatrix} \leq 0, \begin{vmatrix} S_{01} & R \\ S'_{01} & R' \end{vmatrix} \leq 0, \begin{vmatrix} R_{01} & S \\ R'_{01} & S' \end{vmatrix} \leq 0, \begin{vmatrix} S_{01} & S \\ S'_{01} & S' \end{vmatrix} \leq 0. \tag{5.10}$$

Instead of formalizing everything (which can be done with a little care), let us take an example.

Example 5.1

The **Eulerian number** $E(n, k)$, is the number of permutations π from Σ_n with exactly k ascents, i.e. with exactly k places where $\pi_j < \pi_{j+1}$. We know (see, e.g. [16]) that these numbers satisfy the recurrence

$$E(n, k) = (n - k)E(n - 1, k - 1) + (k + 1)E(n - 1, k),$$

with the initial conditions $E(0, k) = \delta_{0k}$, $E(n, 0) = 1$, $n, k \geq 0$. Then (5.4) becomes

$$q(n, k)[n - k + 1 + kq(n - 1, k - 1)] = q(n - 1, k - 1)[n - k + (k + 1)q(n - 1, k)]$$

with $q(n, k) = E(n, k)/E(n, k - 1)$, for $1 \leq k \leq n$. The initial conditions are $q(2, 1) = 1$, $q(3, 1) = 4$, $q(3, 2) = 1/4$ and (we extra define) $q(2, k) = 0$, for $k \geq 2$. We want to prove that $q(n, k) \geq q(n, k + 1)$, for any fixed n and all k. In the sense of the above discussion and notations, here we have $R(n, k) = n - k$, $S(n, k) = k + 1$. Passing to the natural "continuation", (5.7) becomes

$$f[x - y + 1 + yf_{11}] = f_{11}[x - y + (y + 1)f_{10}]. \tag{5.11}$$

In fact, we define the function $f : [2, \infty) \times [1, \infty) \to \mathbb{R}$ first on two shaded strips in Fig. 6 below and then continue by the rule (5.11). In Fig. 6 we indicated the values $f(n, k) = q(n, k)$ for $n \geq 2$, $k \geq 1$ in the lattice nodes. On the vertical walls of the (square) cells Q_2, Q_3, $Q_4, \ldots$, as well as on their lower horizontal walls, we define f to be appropriate linear functions. The upper walls of Q_3, $Q_4, \ldots$, are determined by (5.11). The right walls of Q'_3, $Q'_4, \ldots$ are also determined by (5.11). We fill in f on cells Q_n, Q'_n by appropriate homotopies connecting (possibly nonlinear) functions on the walls.
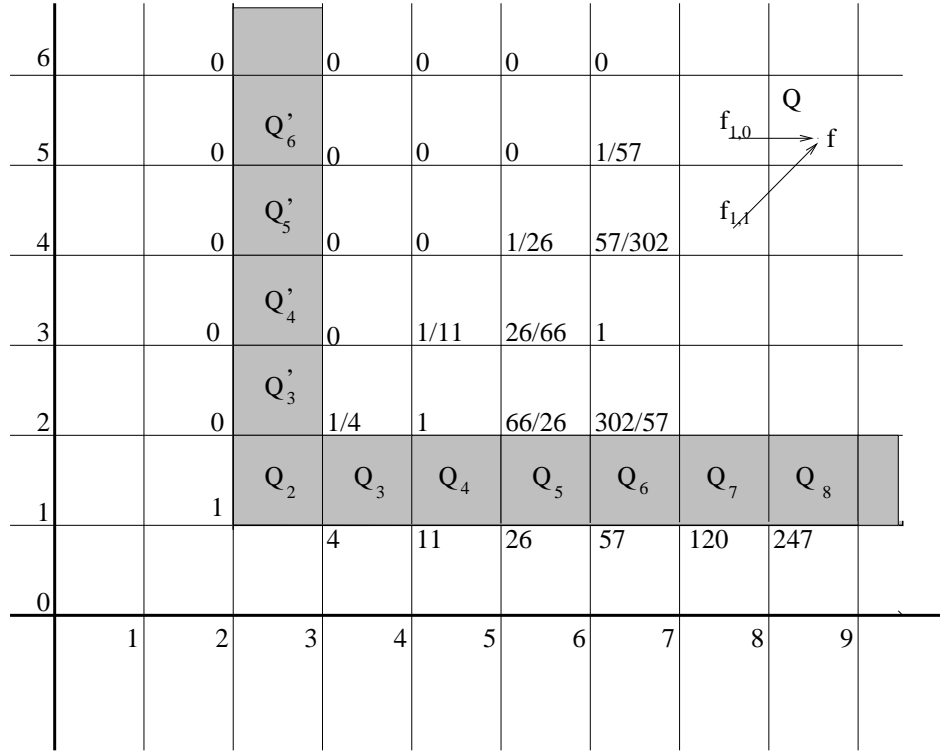
Figure 6: The boundary values of function f

So, let $f(x, 2) = 2^n(x - n) + 2^n - n - 1$, for $x \in [n, n+1]$, $n \geq 2$, $f(2, y) = 2 - y$ for $y \in [1, 2]$ and $f(2, y) = 0$ for $y \geq 2$. Further, let $f(3, y) = \frac{31 - 15y}{4}$ for $y \in [1, 2]$, and, in general, let $f(n, y)$, $y \in [1, 2]$, be the linear function in y between the points $(n, q(n, 1))$ and $(n, q(n, 2))$. The cell Q_2 is surrounded by linear functions $a(x) = 3x - 5$, $x \in [2, 3]$, $b(y) = \frac{31 - 15y}{4}$, $y \in [1, 2]$, $c(x) = \frac{x - 2}{4}$, $x \in [2, 3]$ and

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$d(y) = 2 - y$, $y \in [1, 2]$. Extend f to Q_2 by the homotopy

$$f(x, y) = (2 - y)a(x) + (y - 1)c(x) = \frac{1}{4}(23x + 18y - 11xy - 38), \quad (x, y) \in Q_2.$$

The cell Q_3' is surrounded by functions $a_3'(x) = \frac{x-2}{4}$, $x \in [2, 3]$, $b_3'(y) = \frac{(3-y)^2}{4-y+y(3-y)}$, $y \in [2, 3]$, $c_3'(x) = 0$,

$x \in [2, 3]$ and $d_3'(y) = 0$, $y \in [2, 3]$. Extend f to Q_3' by the homotopy

$$f(x, y) = (3 - x)d_3'(y) + (x - 2)b_3'(y) = \frac{(x-2)(3-y)^2}{4 - y + y(3 - y)}, \quad (x, y) \in Q_3'.$$

Extend f to Q_4', Q_5', ... to be zero. Next, extend f to Q_3, Q_4, ... by the homotopies connecting the

lower walls of Q_n, given by $a_n(x) = 2^n(x - n) + 2^n - n - 1$, for $x \in [n, n + 1]$, $n \geq 2$, and the upper

walls, given by rational functions $c_n(x)$ determined inductively on n by (5.11), thus obtaining $f\big|_{Q_n}$ by

$$f(x, y) = (2 - y)a_n(x) + (y - 1)c_n(x), \quad (x, y) \in Q_n.$$

For example, since $a_2(x) = 3x - 5$, $a_3(x) = 7x - 17$, $c_2(x) = \frac{x-2}{4}$, then

$$c_3(x) = \frac{a_2(x - 1)[x - y + (y + 1)c_2(x - 1)]}{x - y + 1 + ya_2(x - 1)} = \frac{3x - 8}{4},$$

and hence $f\big|_{Q_3}$ is given by

$$f(x, y) = (2 - y)a_3(x) + (y - 1)c_3(x) = \frac{53x + 60y - 25xy - 128}{4}.$$

In this way, f is well defined on the shaded strips on Fig. 6 and extended to $[2, \infty) \times [1, \infty)$ by the

rule (5.11).

It is easy to check that f is continuous and nonnegative and that it is a rational function with no

poles on any open cell, and hence smooth on any open cell. It is also easy to check inductively on n

that f is decreasing in y on Q_n and Q_n' for $n \geq 2$, i.e. $\frac{\partial f}{\partial y}(x, y) \leq 0$ for $(x, y) \in int(Q_n) \cup int(Q_n')$.

Now that we have elaborated carefully the "beginning" of f, the rest is more-or-less automatic. The

inequality (5.9) reduces to $(x - y + 1 + yf_{11})(-1 + f_{10}) \leq (x - y + yf_{10} + f_{10})(-1 + f_{11})$. This is

equivalent to

$$-1 + x f_{10} + f_{10} \leq -f_{10} + x f_{11} + f_{10} f_{11},$$

or, after some rearrangement,

$$f_{11} f_{10} - 2 f_{10} + 1 + x(f_{11} - f_{10}) \geq 0.$$

The second term is non-negative by the induction hypothesis, and the rest is non-negative since

$$f_{11} f_{10} - 2 f_{10} + 1 \geq f_{10}^2 - 2 f_{10} + 1 = (f_{10} - 1)^2 \geq 0.$$

Hence, $\frac{\partial f}{\partial y}(x, y) \leq 0$ for $(x, y) \in int(Q)$, for all Q. So, f is decreasing in y on every open cell, and hence by continuity, f is decreasing in y everywhere. In particular, $q(n, k) = f(n, k) \geq f(n, k+1) = q(n, k+1)$ and we are done.

In the same manner we can prove that $\frac{\partial f}{\partial x}(x - y + 1) \geq f$, hence $\frac{\partial f}{\partial x} \geq 0$, for any $(x, y) \in int(Q)$, for any Q, and this implies $q(n + 1, k) \geq q(n, k)$, for any fixed k and all n. Hence,

$$\frac{E(n+1, k)}{E(n+1, k-1)} \geq \frac{E(n, k)}{E(n, k-1)} \iff \begin{vmatrix} E(n+1, k) & E(n+1, k-1) \\ E(n, k) & E(n, k-1) \end{vmatrix} \geq 0.$$

Apart from settling the "beginning" of f, we can (almost automatically now) prove the well-known log-concave behavior in the second variable of the binomial coefficients (and in general find the log-concave behavior when R and S in (5.1) are constants), q-binomial coefficients, Stirling numbers of the first and second kind, Schläfli numbers, cover many particular results (e.g [20]) and so on.

Of course, the method can be extended in a few ways; for example, to three-or-more term recurrences, recurrences for three or more variables $a(n, k, l)$ etc. but we shall not consider it here.

In a word, a general idea of this method is as follows. Combinatorics gives a recurrence. Pass to the quotients of the neighboring members (in the variable under consideration), pass to the natural

"continuation" f, find some bounds (upper, lower or both, depending on the nature of the problem) of f, make sure that f is differentiable on open cells, examine the rate of growth of f at the "beginning" (i.e. check the sign of the derivative there), and finally, prove inductively from the associated functional equation corresponding to the recurrence, that the sign of the derivative remains the same.

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